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Degenerate Bose System. IV. Quasiparticle Model

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A quasiparticle model of the Bose fluid is developed which provides a simple understanding of the general first-order predictions of the quantum-statistical theory presented in the preceding papers of this series. The essence of the model is that there are two kinds of quasiparticles superimposed on a background sea of nonzero-momentum superfluid particles. It is out of this sea that superfluid quasiparticles, or holes, and normal-fluid quasiparticles are created. The normal fluid is composed entirely of quasiparticles. The results presented in this paper are only valid at temperatures such that the fraction ξ of zero-momentum (superfluid) bosons is appreciable, although $\xi \neq 0$ for all temperatures below the transition temperature inasmuch as Bose-Einstein condensation is assumed to be the underlying cause of the "super" properties of a Bose fluid. At $T=0$, only the zero-momentum bosons and the nonzero-momentum superfluid sea exist, there being no quasiparticles in this case. It is suggested that the excitations in helium II measured by existing neutron scattering experiments correspond to the destruction of superfluid quasiparticles, and that the creation of normal-fluid quasiparticles should also be observable.

1. INTRODUCTION

THE microscopic theory of a degenerate Bose system, or Bose fluid, which is currently accepted is that due to Feynman.[1] It is with a view towards improving the understanding which Feynman's theory has given us that the present series of papers[2] has been undertaken. Our approach in these papers has been to use the formalism of quantum statistics, and in the first two papers a rather general and complicated formalism was developed based on the *assumption* that the lowest, or zero momentum, free-particle state is macroscopically occupied in the Bose fluid. In the third paper the use of the formal theory was illustrated by an application to the model system of a dilute Bose gas of hard spheres. Clearly, London's suggestion[3] that the phenomenon of Bose-Einstein condensation is the underlying microscopic cause of the "super" properties of helium II becomes a basic assumption in our approach.

The present paper is concerned with an initial attempt to derive explicit expressions for the thermodynamic properties and the momentum distribution of a Bose fluid, which can be valid when applied to helium II. The expressions which are here derived are valid only at sufficiently low temperatures such that the fraction ξ of zero-momentum bosons is appreciable. But whereas this initial "first-order" calculation does not permit an investigation of the transition temperature region, it does permit the development of a quasiparticle model for understanding the low-temperature results. In fact, it is within the framework of the quasiparticle model, which we describe in Secs. 2 and 3, that all of the results of this paper are given.

The usefulness of a quasiparticle model is threefold. Firstly, it provides a qualitative picture of the microscopic behavior of the many-body system. Secondly, it makes possible a simple interpretation of the explicit or derived predictions of a microscopic theory. And thirdly, it serves as a basis for making new predictions regarding many-body phenomena such as transport properties, which may not be directly calculable by the microscopic theory. In this paper we are only concerned with the first two of these aspects. For example, in Sec. 5, it is shown that our Bose fluid quasiparticle model can give considerable insight into the quantum-statistical expression which we have derived for the chemical potential.

The requirements on a Bose fluid quasiparticle model are quite severe. It must explain the two-fluid aspect of the Bose fluid, first introduced by Tisza,[4] in a natural way and in such a way that there is no normal fluid at $T=0$. But the model must also permit a large population of nonzero-momentum bosons at $T=0$, a point which has been emphasized by Penrose and Onsager.[5] In fact, these requirements are all satisfied in a natural manner by our quasiparticle model. The essence of the model is that part of the superfluid which is composed of nonzero-momentum bosons behaves as a background "sea," somewhat analogously to the sea of a Fermi fluid. It is out of this sea that superfluid quasiparticles, or holes, and (real) normal fluid quasiparticles are created when $T\neq 0$. The resulting double set of quasiparticles, which

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[1] R. P. Feynman, Phys. Rev. **94**, 262 (1954). See also *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1955), Vol. I, pp. 17–53.

[2] F. Mohling, Phys. Rev. **135**, A831, A855, A876 (1964). Hereafter referred to as MI, MII, and MIII, respectively.

[3] F. London, Nature **141**, 643 (1938).

[4] L. Tisza, Nature **141**, 913 (1938); see also Phys. Rev. **72**, 838 (1947).

[5] O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).

constitute the normal fluid, are characterized not only by their energies $\epsilon_+(\mathbf{p})$ and $\epsilon_-(\mathbf{p})$ and their probability functions $f_+(\mathbf{p})$ and $-f_-(\mathbf{p})$, but also by their "basic" wave-function amplitudes $\alpha_+(\mathbf{p})$ and $\alpha_-(\mathbf{p})$. Thus, the the two different quasiparticles can interfere in the quantum-mechanical sense. The "basic" wave functions referred to are those of the elementary pairs of real bosons, carrying momenta $\mathbf{p}$ and $-\mathbf{p}$, which form the building blocks for the total quasiparticle wave function [see Eqs. (32)–(34)]. In view of this structure of the total wave function, it should be clear that the Bose fluid quasiparticles, although *characterized* by momenta $\mathbf{p}\neq0$, have no momentum in a system at rest.

In order to present the Bose fluid quasiparticle model in the most plausible manner, it has been developed by using the concept of the Bogoliubov transformation.[6] Now, in fact it is shown in Sec. 3 that for $T\neq0$ the complete understanding of the double quasiparticle spectrum requires the *concept* of a double Bogoliubov transformation, although the two transformations are not actually performed. The first Bogoliubov transformation is associated with the separation of the bosons into normal and superfluid components, whereas the second transformation is associated with the separation of the bosons into normal and superfluid quasiparticles. The two transformations are quite different, as the reader will discover.

It is to be emphasized that the quasiparticle model presented in this paper has been developed completely with a view towards explaining the "first-order" predictions of the quantum-statistical theory (Sec. 4). Perhaps second-order calculations will modify some aspects of this model. It may even be that the best interpretation for the first-order results has not been achieved here. But the main point is that the thermodynamic predictions and the momentum distribution [Eq. (30)] given here are results which can be believed, as they are independent of the quasiparticle model.

2. SINGLE BOGOLIUBOV TRANSFORMATION AT $T=0$

The Bogoliubov transformation[6] is the most important feature of the quasiparticle model which will be developed in this paper for the degenerate Bose system. It is therefore of considerable value to review the motivating steps in the derivation of this transformation, introducing at the same time notation which will prove useful to the discussion of the following sections.

We begin by writing the many-body Hamiltonian H in the Fock representation as

$$H=\sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\omega_{\mathbf{k}}+\tfrac{1}{4}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} a_1^\dagger a_2^\dagger\langle\mathbf{k}_1\mathbf{k}_2|V^{(S)}|\mathbf{k}_3\mathbf{k}_4\rangle a_4a_3\,, \quad (1)$$

where

$$\langle\mathbf{k}_1\mathbf{k}_2|V^{(S)}|\mathbf{k}_3\mathbf{k}_4\rangle=\langle\mathbf{k}_1\mathbf{k}_2|V|\mathbf{k}_3\mathbf{k}_4\rangle+\langle\mathbf{k}_1\mathbf{k}_2|V|\mathbf{k}_4\mathbf{k}_3\rangle \quad (2)$$

is a symmetrized matrix element of the basic two-particle interaction. The quantities $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ are the annihilation and creation operators, respectively, of the free bosons, and these satisfy the usual commutation relations. One observes that each of the momentum state sums in (1) includes a contribution from the zero-momentum state, which when considered from the point of view of deviations from the free-particle condition may be macroscopically occupied (in a system at rest). In order to investigate the ground state ($T=0$) of the Bose system, it is therefore necessary to treat the zero-momentum state specially.

The difficulty presented by the zero-momentum state can be most easily seen when attempting to calculate the deviations from the free Bose-gas condition by applying many-body perturbation theory in a straightforward manner. One assumes in first approximation that the operators a_0 and $a_0^\dagger$ may each be replaced by the number $\langle N_0\rangle^{1/2}$,[7] and one then separates from the interaction part of the Hamiltonian (1) the resulting diagonal terms. Finally, a detailed study[8] of the perturbation treatment of the off-diagonal interaction terms shows that the perturbation theory for the ground-state energy diverges in the limit $\langle N\rangle, \Omega\to\infty$ (keeping $n=\langle N\rangle/\Omega$ finite). It is the macroscopic occupation of the zero-momentum state, i.e., the fact that $\langle N_0\rangle\sim\langle N\rangle$, which causes this divergence. The Bogoliubov transformation is a method whereby this failure of perturbation theory may be overcome.

According to the preceding paragraph, the Hamiltonian (1) can be approximately rewritten for the degenerate Bose system as

$$H\cong H_0+H'\,, \quad (3)$$

where

$$\begin{aligned}H_0\equiv&\sum_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\omega_{\mathbf{p}}+\tfrac{1}{4}\langle N_0\rangle^2\langle00|V^{(S)}|00\rangle\\&+\langle N_0\rangle\sum_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\langle\mathbf{p}0|V^{(S)}|\mathbf{p}0\rangle\\&+\tfrac{1}{4}\langle N_0\rangle\sum_{\mathbf{p}}[a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger\langle\mathbf{p}-\mathbf{p}|V^{(S)}|00\rangle\\&\qquad+\langle00|V^{(S)}|\mathbf{p}-\mathbf{p}\rangle a_{\mathbf{p}}a_{-\mathbf{p}}]\,,\end{aligned} \quad (4)$$

$$\begin{aligned}H'\equiv&\tfrac{1}{2}\langle N_0\rangle^{1/2}\sum_{\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4} a_2^\dagger\langle0\mathbf{p}_2|V^S|\mathbf{p}_3\mathbf{p}_4\rangle a_3a_4\\&+\tfrac{1}{2}\langle N_0\rangle^{1/2}\sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3} a_1^\dagger a_2^\dagger\langle\mathbf{p}_1\mathbf{p}_2|V^{(S)}|\mathbf{p}_30\rangle a_3\\&+\tfrac{1}{4}\sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4} a_1^\dagger a_2^\dagger\langle\mathbf{p}_1\mathbf{p}_2|V^{(S)}|\mathbf{p}_3\mathbf{p}_4\rangle a_3a_4\,,\end{aligned} \quad (5)$$

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[6] N. Bogoliubov, J. Phys. (USSR) **11**, 23 (1947). See also S. T. Beliaev, *The Many-Body Problem*, edited by C. DeWitt (John Wiley & Sons, Inc., New York, 1959), pp. 343–356.

[7] This step is analogous to the introduction of the x-ensemble formulation of quantum statistics in T. D. Lee and C. N. Yang [Phys. Rev. **117**, 897 (1960)]. In the x-ensemble formalism, however, the dynamical (as opposed to the statistical) quantum-mechanical effects of the operators a_0 and $a_0\dagger$ are included (see MI).

[8] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 135 (1957).

and where we have introduced the convention that $\mathbf{k}_i \to \mathbf{p}_i$ when $\mathbf{k}_i$ cannot be zero. In this section we shall only consider the case of weak interactions and in this case the off-diagonal interaction terms which give the dominant contribution to the ground-state energy are those which have been included with H_0 in (4).[8]

The approximate Hamiltonian (3) is a function of the variable number N_0, and therefore the number of nonzero-momentum particles is not a conserved quantity. In statistical mechanics, when we wish to consider the possibility that the total particle number may have fluctuations we use the grand canonical density matrix

$$\rho_G = \exp[-\Omega f] \exp\beta[G-H], \tag{6a}$$

instead of the canonical density matrix

$$\rho = \exp\beta[\Psi - H], \tag{6b}$$

where f is the grand potential, G is the Gibbs free-energy or thermodynamic potential, Ψ is the Helmholtz free energy, and $\beta = (kT)^{-1}$. It seems natural, therefore, to consider instead of the Hamiltonian H of (3) the Hamiltonian[9]

$$\begin{aligned} H_G &\equiv H_0 - g \sum_p a_\mathbf{p}^\dagger a_\mathbf{p} + H' \\ &= H_0' + H', \end{aligned} \tag{7}$$

where

$$\begin{aligned} H_0' = \sum_\mathbf{p} a_\mathbf{p}^\dagger a_\mathbf{p} \epsilon_\mathbf{p} &+ \tfrac{1}{2}\langle N\rangle \xi^2 W(00|00) \\ &+ 2\xi \sum_\mathbf{p} a_\mathbf{p}^\dagger a_\mathbf{p} W(\mathbf{p}0|\mathbf{p}0) \\ &+ \tfrac{1}{2}\xi \sum_\mathbf{p} [a_\mathbf{p}^\dagger a_{-\mathbf{p}}^\dagger W(\mathbf{p}-\mathbf{p}|00) \\ &\qquad + W(00|\mathbf{p}-\mathbf{p}) a_\mathbf{p} a_{-\mathbf{p}}], \end{aligned} \tag{8}$$

$$W(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4) \equiv \tfrac{1}{2}\langle N\rangle \langle \mathbf{k}_1\mathbf{k}_2| V^{(S)} |\mathbf{k}_3\mathbf{k}_4\rangle, \tag{9}$$

$$\xi = \langle N_0\rangle / \langle N\rangle, \tag{10}$$

$$\epsilon_\mathbf{p} \equiv \omega_\mathbf{p} - g, \tag{11}$$

and g is the chemical potential or thermodynamic potential per particle of the system. Thus, in the Hamiltonian (7) all single-particle energies are expressed relative to the chemical potential g. Clearly then, we may eventually derive an expression for g by interpreting it as the self-energy of a zero-momentum boson. The quantity ξ of (10) is the fraction of zero-momentum bosons, a number which is probably ~ 0.1 even at $T=0$,[5] but never zero below the transition temperature of the degenerate Bose system. In fact, if the interpretation given by London[3] is correct that the transition from He I to He II can be characterized by the onset of Bose-Einstein condensation, then the transition temperature may be calculated by setting $\xi = 0^+$. Finally, we have introduced the function $W(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)$ in Eq. (8) because it is well-defined in the limit $\langle N\rangle$, $\Omega \to \infty$.

Our next step is to diagonalize the Hamiltonian (8), and it is at this point that we introduce the Bogoliubov transformation,[6]

$$\begin{aligned} \xi_\mathbf{p} &= F_+^{1/2}(\mathbf{p}) a_\mathbf{p} + F_-^{1/2} a_{-\mathbf{p}}^\dagger, \\ \xi_\mathbf{p}^\dagger &= F_+^{1/2}(\mathbf{p}) a_\mathbf{p}^\dagger + F_-^{1/2} a_{-\mathbf{p}}, \end{aligned} \tag{12}$$

where we assume that the coefficients $F_+^{1/2}$ and $F_-^{1/2}$ are real quantities. In order that the transformation be canonical, i.e., that the $\xi_\mathbf{p}$ and $\xi_\mathbf{p}^\dagger$ satisfy the same commutation relations as the corresponding free boson operators, we must set

$$F_+(\mathbf{p}) - F_-(\mathbf{p}) = 1. \tag{13}$$

Upon inverting the transformation equations (12) and substituting the resulting expressions into Eq. (8), one obtains for H_0' the result

$$\begin{aligned} H_0' = \tfrac{1}{2}\langle N\rangle \xi^2 W(00|00) &+ \tfrac{1}{2} \sum_p [\epsilon_+(\mathbf{p}) + \epsilon_1(\mathbf{p})] \xi_\mathbf{p}^\dagger \xi_\mathbf{p} \\ &- \tfrac{1}{2} \sum_\mathbf{p} [\epsilon_-(\mathbf{p}) + \epsilon_1(\mathbf{p})] \xi_\mathbf{p} \xi_\mathbf{p}^\dagger \\ &+ \sum_\mathbf{p} [\tfrac{1}{2}\xi W(\mathbf{p}-\mathbf{p}|00) F_+(\mathbf{p}) \\ &+ \tfrac{1}{2}\xi W(00|\mathbf{p}-\mathbf{p}) F_-(\mathbf{p}) - \epsilon_1(\mathbf{p}) F_+^{1/2}(\mathbf{p}) F_-^{1/2}(\mathbf{p})] \\ &\qquad \times [\xi_\mathbf{p}\xi_\mathbf{p} + \xi_\mathbf{p}^\dagger \xi_{-\mathbf{p}}^\dagger], \end{aligned} \tag{14}$$

where

$$\epsilon_1(\mathbf{p}) = \epsilon(\mathbf{p}) + 2\xi W(\mathbf{p}0|\mathbf{p}0), \tag{15}$$

$$\begin{aligned} \epsilon_+(\mathbf{p}) = -\epsilon_-(\mathbf{p}) &\equiv [F_+(\mathbf{p}) + F_-(\mathbf{p})]\epsilon_1(\mathbf{p}) - \xi F_+^{1/2}(\mathbf{p}) F_-^{1/2}(\mathbf{p}) \\ &\quad \times [W(\mathbf{p}-\mathbf{p}|00) + W(00|\mathbf{p}-\mathbf{p})]. \end{aligned} \tag{16}$$

The condition of diagonalization, which involves setting each of the coefficients in the last term of (14) equal to zero, then gives a second relation between $F_+(\mathbf{p})$ and $F_-(\mathbf{p})$,

$$\begin{aligned} \tfrac{1}{2}\xi W(00|\mathbf{p}-\mathbf{p})[F_+(\mathbf{p}) + F_-(\mathbf{p})] \\ - \epsilon_1(\mathbf{p}) F_+^{1/2}(\mathbf{p}) F_-^{1/2}(\mathbf{p}) = 0, \end{aligned} \tag{17}$$

where we have used the Hermitian property of the two-particle interaction V.

It is convenient to solve for $F_+(\mathbf{p})$ and $F_-(\mathbf{p})$ by writing them in the form [see Eq. (13)]

$$F_+(\mathbf{p}) = [1-\alpha^2(\mathbf{p})]^{-1}; \quad F_-(\mathbf{p}) = \alpha^2(\mathbf{p})[1-\alpha^2(\mathbf{p})]^{-1}. \tag{18}$$

One then finds from Eqs. (15)–(17) the expressions

$$\alpha(\mathbf{p}) = \xi W(00|\mathbf{p}-\mathbf{p})[\epsilon_+(\mathbf{p}) + \epsilon_1(\mathbf{p})]^{-1}, \tag{19}$$

$$\begin{aligned} \epsilon_+(\mathbf{p}) &= -\epsilon_-(\mathbf{p}) \\ &= [\epsilon_1^2(\mathbf{p}) - \xi^2 W^2(00|\mathbf{p}-\mathbf{p})]^{1/2} \underset{\mathbf{p}\to 0}{\longrightarrow} O(\mathbf{p}). \end{aligned} \tag{20}$$

The Bogoliubov transformation which we have just presented transforms the $T=0$ degenerate Bose system

[9] See also N. Hugenholtz and D. Pines, Phys. Rev. 116, 489 (1959).

from a free-particle description to a quasiparticle description. We interpret the three resulting single-quasiparticle energies $\epsilon_1(\mathbf{p})$, $\epsilon_+(\mathbf{p})$, and $\epsilon_-(\mathbf{p})$ as follows:

$$\begin{aligned}\epsilon_1(\mathbf{p}) &= \text{internal energy of a quasiparticle (within the Bose fluid)},\\ \epsilon_+(\mathbf{p}) &= \text{kinetic energy of a normal-fluid quasiparticle (within the Bose fluid)},\\ \epsilon_-(\mathbf{p}) &= \text{kinetic energy of a superfluid quasiparticle (within the Bose fluid)}.\end{aligned} \tag{21}$$

With these interpretations we have introduced the two-fluid aspect of a degenerate Bose system, first stated by Tisza.[4] The justification for these interpretations arises after first defining the normal-fluid vacuum state by the equations

$$|0\rangle = \prod_{\mathbf{p}>0} |0_{\mathbf{p}}\rangle, \quad \xi_{\mathbf{p}}|0_{\mathbf{p}}\rangle = 0, \tag{22}$$

where the product over all $\mathbf{p}>0$ means that each momentum pair $(\mathbf{p}, -\mathbf{p})$ is to be considered only once. The ground-state energy is next readily obtained by writing the Hamiltonian (14), together with the diagonalization condition (17), in the form

$$H_0' = \tfrac{1}{2}\langle N\rangle \xi^2 W(00|00) - \tfrac{1}{2}\sum_p [\epsilon_-(\mathbf{p})+\epsilon_1(\mathbf{p})] + \tfrac{1}{2}\sum_{\mathbf{p}} [\epsilon_+(\mathbf{p})-\epsilon_-(\mathbf{p})]\xi_{\mathbf{p}}^\dagger \xi_{\mathbf{p}}. \tag{23}$$

Now, at $T=0$ the degenerate Bose system must be entirely superfluid, if by *superfluid* we mean that portion of the Bose system which is without entropy. Therefore we identify the energy $[\epsilon_-(\mathbf{p})+\epsilon_1(\mathbf{p})]$ with a superfluid quasiparticle of momentum $\mathbf{p}$ and the energy $[\epsilon_+(\mathbf{p})+\epsilon_1(\mathbf{p})]$ with a normal-fluid quasiparticle of momentum $\mathbf{p}$, since at $T=0$ the last term of (23) does not contribute. We then assume that only the energies $\epsilon_+(\mathbf{p})$ and $\epsilon_-(\mathbf{p})$ are relevant for describing the normal and superfluid quasiparticle energetics *within* the Bose fluid, and thereby we arrive at the interpretations (21). Further justification can only be obtained when the $T\neq 0$ considerations of Sec. 3 are compared with the predictions of quantum statistics in Sec. 4.

The above discussion shows how the two-fluid model of a degenerate Bose fluid can arise in a natural way in a microscopic theory. Moreover, the interpretation which is made is such that the superfluid portion of the Bose fluid contains nonzero as well as zero-momentum bosons,[10] the latter probably always being in the minority[5] in real helium II. Corresponding to *each* of the two fluids is a set of quantum-mechanical normal modes, or quasiparticles. Of course, at $T=0$ only one fluid, the superfluid, occurs and this fact is probably why the above interpretation of the Bogoliubov transformation has been previously overlooked in the literature. In fact, the above interpretation is still not completely correct as will be shown in the following section when the case $T\neq 0$ is considered. It should be observed at this point, however, that a "double" quasiparticle spectrum is completely expectable. For the Bogoliubov transformation (12) is such that a normal-fluid state vector of momentum $\mathbf{p}$ may be interpreted as a combination of amplitudes for boson pairs with each boson pair carrying the momenta $\mathbf{p}$ and $-\mathbf{p}$ [see Eq. (26)]. Therefore, in order to fully utilize the free-particle coordinates available in such a canonical transformation one must have two different kinds of quasiparticles for each momentum pair $(\mathbf{p}, -\mathbf{p})$. We note that one consequence of this pairing is that all quasiparticles in a system at rest have momentum zero. As a final remark concerning the double-quasiparticle spectrum, it is important to observe that although $\epsilon_+(\mathbf{p}) = -\epsilon_-(\mathbf{p})$ in the weak-interaction limit of this section, this simple relation between $\epsilon_+(\mathbf{p})$ and $\epsilon_-(\mathbf{p})$ is not true in general except perhaps for low-momentum values.

In order to compare the prediction of Eq. (23) for the ground-state energy of a system with that given by more sophisticated theories, in which the interactions may be strong, an important modification is necessary. To see how this modification arises, we consider the asymptotic form of the superfluid energy $[\epsilon_-(\mathbf{p})+\epsilon_1(\mathbf{p})]$,

$$[\epsilon_-(\mathbf{p})+\epsilon_1(\mathbf{p})] \xrightarrow[\mathbf{p}\to\infty]{} \tfrac{1}{2}\xi^2 W^2(00|\mathbf{p}-\mathbf{p})\epsilon^{-1}(\mathbf{p}). \tag{24}$$

Suppose we assume that the function $W(00|\mathbf{p}-\mathbf{p})$ has no natural cutoff when $\mathbf{p}\to\infty$. Then, in this case the second term of (23) would be a divergent integral in the limit $\Omega\to\infty$. Subtracting the "divergent" term given by $-\frac{1}{2}\sum_{\mathbf{p}}$ times the right-hand side of (24), we would obtain the "well-defined" expression for the Hamiltonian

$$\begin{aligned} H_0'' &= H_0' + \tfrac{1}{4}\sum_{\mathbf{p}} \xi^2 W^2(00|\mathbf{p}-\mathbf{p})\epsilon^{-1}(\mathbf{p}) \\ &= \tfrac{1}{2}\langle N\rangle \xi^2 W(00|00) \\ &\quad - \tfrac{1}{2}\sum_p \{[\epsilon_-(\mathbf{p})+\epsilon_1(\mathbf{p})] - \tfrac{1}{2}\xi^2 W^2(00|\mathbf{p}-\mathbf{p})\epsilon^{-1}(\mathbf{p})\} \\ &\quad + \tfrac{1}{2}\sum_{\mathbf{p}} [\epsilon_+(\mathbf{p})-\epsilon_-(\mathbf{p})]\xi_{\mathbf{p}}^\dagger\xi_{\mathbf{p}}. \end{aligned} \tag{25}$$

In fact, this last Hamiltonian has a very close correspondence with the general $(T\neq 0)$ expression for the Helmholtz free energy which is written down in the next section. It is also essentially the effective Hamiltonian which one encounters when the pseudopotential method is applied to a low-density gas of Bose hard spheres at $T=0$.[8] We shall call the second term of (25) the *Bogoliubov transformation energy* (at $T=0$).

The ground-state vector $|0_{\mathbf{p}}\rangle$ of (22) can be expressed in terms of the corresponding free boson vacuum state vector $|\,\rangle_{\mathbf{p}}$ by using the transformation equations (12).

[10] We are indebted to Professor C. N. Yang for emphasizing this point to us (private communication).

A straightforward calculation yields the result

$$|0_{\mathbf{p}}\rangle=[1-\alpha^2(\mathbf{p})]^{1/2}\exp[-\alpha(\mathbf{p})a_{\mathbf{p}}^{\dagger}a_{-\mathbf{p}}^{\dagger}]|\ \rangle_{\mathbf{p}}. \quad (26)$$

From this expression one sees explicitly how the free Boson pairs occur in the superfluid at $T=0$. Their momentum distribution is given by the expression

$$\langle n(\mathbf{p})\rangle \underset{T=0}{\longrightarrow} \langle 0_{\mathbf{p}}|a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}}|0_{\mathbf{p}}\rangle=F_-(\mathbf{p}), \quad (27)$$

where $F_-(\mathbf{p})$ is given by Eq. (18).

3. DOUBLE BOGOLIUBOV TRANSFORMATION AT $T\neq 0$

An important characteristic of the many-body system is that its microscopic description cannot in general be given in terms of a "pure" quantum-mechancial state. Because of our lack of knowledge concerning the time development of many-body systems we must describe them by the "mixed" states which result from statistical averaging. Since our purpose in this paper is to describe in simple terms the predictions of quantum-statistical averaging for the degenerate Bose fluid, we shall be forced to use the concept of a mixed wave function in this section. By a mixed wave function or state vector we mean one which properly gives the distribution functions and thermodynamic properties of the system. Of course, at the special temperature $T=0$ the many-body system can be described by a pure state without statistical averaging, as we have done in Sec. 2.

The Bogoliubov transformation equations (12) and (13),

$$\xi_{\mathbf{p}}=F_+^{1/2}(\mathbf{p})a_{\mathbf{p}}+F_-^{1/2}(\mathbf{p})a_{-\mathbf{p}}^{\dagger}, \qquad F_+(\mathbf{p})-F_-(\mathbf{p})=1, \quad (28)$$

will henceforth be called the first Bogoliubov transformation equations. The "application" of this first transformation to a Bose fluid at a general temperature $T\neq 0$ will be interpreted as meaning that the real bosons are separated by the transformation into normal and superfluid components. We have seen with Eq. (27) that the quantity $F_-(\mathbf{p})$ is the momentum distribution of the superfluid nonzero-momentum bosons at $T=0$. This fact suggests that we write for $T\neq 0$

$F_-(\mathbf{p})=$ number of superfluid bosons of momentum $\mathbf{p}$ per free-particle state,

$F_+(\mathbf{p})=$ number of normal-fluid bosons of momentum $\mathbf{p}$ per quasiparticle state, (29)

$\nu'(\mathbf{p})=$ number of quasiparticles of momentum $\mathbf{p}$ per free-particle state,

where a free-particle state is one such that the volume element in momentum space is given by $\Omega\delta^3\mathbf{p}=1$. With the admittedly-peculiar identifications (29) we may immediately write for the momentum distribution the general expression

$$\begin{aligned}\langle n(\mathbf{p})\rangle&=F_-(\mathbf{p})+\nu'(\mathbf{p})F_+(\mathbf{p})\\&=[1-\alpha^2(\mathbf{p})]^{-1}[\alpha^2(\mathbf{p})+\nu'(\mathbf{p})],\end{aligned} \quad (30)$$

where we shall continue to use Eqs. (18) for $F_+(\mathbf{p})$ and $F_-(\mathbf{p})$. Both of the quantities $\alpha(\mathbf{p})$ and $\nu'(\mathbf{p})$ will be determined below for $T\neq 0$.

Our understanding of the degenerate Bose system in Sec. 2 was that its quasiparticles were of two types, corresponding to the normal and superfluids, respectively. And yet in Eq. (30) we have associated the quasiparticles entirely with the normal-fluid momentum distribution. The underlying reason for this is the important assumption which we shall now make that *no* quasiparticles exist at $T=0$. The superfluid is *without* quasiparticles at $T=0$. We may correlate this assumption with the need to have two kinds of quasiparticles by assuming that *the superfluid quasiparticles are holes in the nonzero-momentum superfluid*. The superfluid quasiparticles do not therefore contribute to the superfluid-momentum distribution, but rather their effect on $\langle n(\mathbf{p})\rangle$ is implicitly contained in $\nu'(\mathbf{p})$ which must be multiplied by $F_+(\mathbf{p})>1$ in order to give the normal-fluid momentum distribution. Having presented this concept for what is meant by quasiparticles in the Bose fluid, we shall continue to use the single-quasiparticle energy interpretations given by (21). It is interesting to note that Lieb[11] has arrived at a similar interpretation of the one-dimensional Bose gas model which he has studied.

At this point we can write down a "mixed" state vector $|T\rangle$ which properly reproduces the momentum distribution (30). Upon comparing Eqs. (18), (26), and (27) with Eq. (30), one can readily verify that $|T\rangle$ is given by

$$|T\rangle=\prod_{\mathbf{p}>0}\{[1+\alpha^{-2}(\mathbf{p})\nu'(\mathbf{p})]^{1/2}|0_p\rangle-C_0(\mathbf{p})|\ \rangle_p\}$$

$$C_0(\mathbf{p})=\{[1+\alpha^{-2}(p)\nu'(\mathbf{p})]^{1/2}[1-\alpha^2(\mathbf{p})]^{1/2} -[1-\alpha^2(\mathbf{p})-\nu'(\mathbf{p})]^{1/2}\}>0, \quad (31)$$

where the second term in $|T\rangle$ is required in order that $|T\rangle$ be normalized to unity. Physically, we can understand why $|T\rangle$ is a linear combination of free-particle and quasiparticle vacuum states when $T\neq 0$, because the free-boson vacuum state $|\ \rangle$ is depleted by excitations at nonzero temperatures. Since $|T\rangle$ is a "mixed" state it cannot be an eigenstate of $\xi_{\mathbf{p}}^{\dagger}\xi_{\mathbf{p}}$ or $\sum_{\mathbf{p}}\xi_{\mathbf{p}}^{\dagger}\xi_{\mathbf{p}}$, so that the the form of Eq. (31) is not inconsistent with other considerations.

Lee, Huang, and Yang[8] have shown that the N boson position space wave function corresponding to $|T\rangle$ has

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[11] Elliot H. Lieb and Werner Liniger, Phys. Rev. **130**, 1605 (1963); Elliot H. Lieb, *ibid.* **130**, 1616 (1963).

the form (for $N=$even)

$$\Phi_T(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N)=\Omega^{-N/2}\prod_{\mathbf{p}>0}[1-\alpha^2(\mathbf{p})-\nu'(\mathbf{p})]^{1/2}+\sum_{n=1}^{N/2}\chi_n(\mathbf{r}_1\mathbf{r}_2\cdots\mathbf{r}_N). \quad (32)$$

For large n values, the second free-boson vacuum term can be neglected in (31) and in this case the expression for χ_n is

$$\chi_n \xrightarrow[N\geqslant 2n\gg 1]{} C_T\Omega^{-N/2}[(N!)^{-1}(N-2n)!]^{1/2}\times N^n\sum_{\{\mathbf{r}_i\}}f(\mathbf{r}_{12})f(\mathbf{r}_{34})\cdots f(\mathbf{r}_{2N-1,2N}), \quad (33)$$

$$C_T=\prod_{p>0}[1+\alpha^{-2}(\mathbf{p})\nu'(\mathbf{p})]^{1/2}[1-\alpha^2(\mathbf{p})]^{1/2},$$

where

$$f(\mathbf{r})=-N^{-1}\sum_p e^{i\mathbf{p}\cdot\mathbf{r}}\alpha(\mathbf{p}), \quad (34)$$

and the $\sum$ over $\{\mathbf{r}_i\}$ is a sum over the $N![(N-2n)!n!2^n]^{-1}$ different ways of selecting n pairs of $2n$ bosons from among a total of N bosons. In deriving (34) we have assumed that $\alpha(-\mathbf{p})=\alpha(\mathbf{p})$, which will be true for an isotropic system. We see from (34) that $\alpha(\mathbf{p})$ is essentially the amplitude for relative two-boson plane-wave states in the Bose fluid. It is therefore a most important function of the microscopic quasiparticle theory. We shall call the function $f(\mathbf{r})$ the *basic pair wave function.*

We have not yet clarified the microscopic understanding of the "double" quasiparticle picture at $T\neq 0$, because the "first" Bogoliubov transformation (28) only separates the real bosons into two different components. The clarification is achieved by introducing a *second* Bogoliubov transformation,

$$\begin{aligned}\xi_{\mathbf{p}}'&=f_+^{1/2}(\mathbf{p})a_{\mathbf{p}}+f_-^{1/2}(\mathbf{p})a_{-\mathbf{p}}^\dagger,\\ f_+(\mathbf{p})&-f_-(\mathbf{p})\cong 1.\end{aligned} \quad (35)$$

Whereas the first Bogoliubov transformation is concerned with the separation of the bosons into the normal and superfluid components, the second Bogoliubov transformation is concerned with the separation of the bosons into normal and superfluid quasiparticles. Thus, the first Bogoliubov transformation is macroscopically oriented while the second one is microscopically oriented, but both are transformations away from the free-boson description of the Bose fluid. The fact that (f_+-f_-) may be only approximately equal to unity is a result of the quantum-statistical theory discussed in Sec. 4 where explicit expressions for $f_+(\mathbf{p})$ and $f_-(\mathbf{p})$ are given. It is possible that an "exact" calculation would give $f_+-f_-\equiv 1$, but it is more likely that the transformation is not completely canonical. That is to say, the normal and superfluid quasiparticles are not independent quantities and their amplitudes can interfere.

The interpretation which we give to the quantities $f_+(\mathbf{p})$ and $f_-(\mathbf{p})$ is as follows:

$-f_-(\mathbf{p})=$ fraction of bosons per superfluid quasiparticle in the absence of temperature,
$f_+(\mathbf{p})=$ fraction of bosons per normal-fluid quasiparticle in the absence of temperature, (36)

where the minus sign in the first expression is required because of our interpretation that superfluid quasiparticles are really holes in the superfluid "sea." The effect of temperature is that *in the absence of statistics* the quasiparticles are distributed according to the following weighting functions [see Eqs. (21)]:

$P_-(\mathbf{p})\equiv -f_-^{-1}(\mathbf{p})e^{-\beta\epsilon_-(\mathbf{p})}$
$=$ probability for finding a superfluid quasiparticle of momentum $\mathbf{p}$ in a single free-boson state,
$P_+(\mathbf{p})\equiv f_+^{-1}(\mathbf{p})e^{-\beta\epsilon_+(\mathbf{p})}$
$=$ probability for finding a normal-fluid quasiparticle of momentum $\mathbf{p}$ in a single free-boson state. (37)

Because the normal and superfluid quasiparticles are *not* independent quantities, whereas the real bosons are, the total probability P in the absence of statistics for finding either quasiparticle in a single free-boson state is obtained by taking the inverse of $(P_+^{-1}+P_-^{-1})$.

$$P\equiv(P_+^{-1}+P_-^{-1})^{-1}=[f_+e^{\beta\epsilon_+}-f_-e^{\beta\epsilon_-}]^{-1}. \quad (38)$$

The effect of statistics on this last result is then to give us the quantity $\nu'(\mathbf{p})$ of (29) by the expression

$$\begin{aligned}\nu'(\mathbf{p})&=P(1-P)^{-1}\\&=[f_+(\mathbf{p})e^{\beta\epsilon_+(\mathbf{p})}-f_-(\mathbf{p})e^{\beta\epsilon_-(\mathbf{p})}-1]^{-1}.\end{aligned} \quad (39)$$

It is to be emphasized that in the derivation of (39) one must recognize that it is only the free bosons which satisfy Bose statistics, and not the quasiparticles. This is the motivating reason for the several steps (36)–(38) which precede the result (39).

We now use the above concepts to derive an expression for the amplitude $\alpha(\mathbf{p})$ of Eqs. (18) and (34). For this purpose we first introduce the "microscopic" amplitudes $\alpha_+(\mathbf{p})$ and $\alpha_-(\mathbf{p})$ by the definitions [see Eq. (19)]

$$\alpha_\pm(\mathbf{p})=\xi W_{0,0}(00|\mathbf{p},-\mathbf{p})[\epsilon_\mp(\mathbf{p})+\epsilon_1(\mathbf{p})]^{-1}, \quad (40)$$

where $W_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)$ for i and j belonging to the set $(+,-,0,1)$ is the strong interaction generalization of Eq. (9) to be defined in Sec. 4. The relation between the partial amplitudes (40) and the total amplitude is then determined by the equation

$$P_+P_-\alpha(\mathbf{p})=\nu'(\mathbf{p})[P_-\alpha_-(\mathbf{p})+P_+\alpha_+(\mathbf{p})], \quad (41)$$

which upon substituting Eqs. (37) becomes

$$\alpha(\mathbf{p})=\nu'(\mathbf{p})\{\alpha_-(\mathbf{p})f_+(\mathbf{p})e^{\beta\epsilon_+(\mathbf{p})}-\alpha_+(\mathbf{p})f_-(\mathbf{p})e^{\beta\epsilon_-(\mathbf{p})}\xrightarrow[\beta\to\infty]{}\alpha_-(\mathbf{p}). \quad (42)$$

It can be seen from Eqs. (39) and (42) that the second Bogoliubov transformation (35) has enabled us to increase our microscopic understanding of the quasiparticle distribution function $\nu'(\mathbf{p})$ and the amplitude $\alpha(\mathbf{p})$. Actually, we never apply the two Bogoliubov transformations to explicitly calculate the properties of a Bose fluid, as indeed we probably cannot at $T\neq 0$ where the mixed states due to statistical averaging occur. Rather, the two Bogoliubov transformations are invoked in this section in order to facilitate our understanding of the results which are predicted by the quantum-statistical theory as discussed in Sec. 4. We note that at $T=0$ the double Bogoliubov transformation becomes a single Bogoliubov transformation because there are no quasiparticles to be described at $T=0$.

We next turn our attention to the calculation of the thermodynamic properties of a Bose fluid. In Sec. 2 we introduced the chemical potential g into the Hamiltonian (7) in order to effectively deal with the fact that the total number of nonzero-momentum bosons is not conserved for a Bose fluid. But, as we shall show in Sec. 5, the chemical potential can be calculated as the self-energy of a zero-momentum particle (which exerts no pressure and carries no entropy). This means that we may choose for thermodynamic independent variables the set β, n, and Ω, where $n=\langle N\rangle/\Omega$ is the total density of bosons. For this set of variables, the appropriate characteristic function of thermodynamics is the Helmholtz free energy Ψ.

The relation between the Helmholtz free energy and the average energy $\langle E\rangle$ is given by the familiar expression

$$\Psi=\beta^{-1}\int_0^\beta dt\langle E(t)\rangle. \tag{43}$$

Moreover, for a free Bose gas below the transition temperature the Helmholtz free energy is given by

$$\Psi_0=(1-\xi)\langle N\rangle g_0-\beta^{-1}\textstyle\sum_{\mathbf{p}}\ln[1+\nu(\mathbf{p})], \tag{44}$$

where $\nu(\mathbf{p})$ is the momentum distribution for the free bosons, ξ is the fractional occupation of the zero-momentum state, and $g_0\sim\langle N\rangle^{-1}$. Upon comparing these last two expressions with the approximate Hamiltonian (25) [and remembering Eq. (7)], we arrive at the following plausible expression for Ψ:

$$\Psi(n,\beta,\Omega)=(1-\xi)\langle N\rangle g+\tfrac{1}{2}\xi^2\langle N\rangle W_{0,0}(00|00)+\beta^{-1}\int_0^\beta dtE_B(t)-\beta^{-1}\sum_{\mathbf{p}}\{\ln[1+\nu'(\mathbf{p})]+\tfrac{1}{2}\ln[1+F_-(\mathbf{p})]\}, \tag{45}$$

where $W_{0,0}$ is the strong interaction generalization of W and $E_B(t)$ is the Bogoliubov transformation energy at $T\neq 0$. The second $F_-(\mathbf{p})$ part of the last term is the contribution of the momentum distribution of the superfluid sea [see Eqs. (29)] which occurs in the generalization of the second term of (44). This last term in (45) can be viewed as the result of the quantum-statistical averaging of the last term in Eq. (25). The quantum-statistical theory also includes strong interaction contributions from H', Eq. (5), in a manner which would be difficult to clarify here, but which corresponds to the fact that $W_{ij}\neq W$.

The Bogoliubov transformation energy is calculated in analogy with Eq. (41), except that the effect of statistics is not considered. Thus, one obtains

$$\begin{aligned}E_B(t)=&-\tfrac{1}{2}\sum_{\mathbf{p}}\{(P_++P_-)^{-1}(P_-{}^t[\epsilon_-(\mathbf{p})+\epsilon_1(\mathbf{p})]+P_+{}^t[\epsilon_+(\mathbf{p})+\epsilon_1(\mathbf{p})])-\tfrac{1}{2}\xi^2W_{0,0}(00|\mathbf{p},-\mathbf{p})\epsilon^{-1}(\mathbf{p})\}\\&\xrightarrow[\beta\to\infty]{}-\tfrac{1}{2}\sum_{\mathbf{p}}\{[\epsilon_-(\mathbf{p})+\epsilon_1(\mathbf{p})]-\tfrac{1}{2}\xi^2W_{0,0}{}^2(00|\mathbf{p},-\mathbf{p})\epsilon^{-1}(p)\},\end{aligned} \tag{46}$$

where

$$P_\pm{}^{(+)}(\mathbf{p})=\pm f_\pm(\mathbf{p})\exp[-t\epsilon_\pm(\mathbf{p})].$$

The determination of thermodynamic quantities from the Helmholtz free energy is accomplished with the aid of well-known formulas. Thus, the average energy is given by

$$\langle E\rangle=(\partial/\partial\beta)(\beta\Psi)_{\Omega,n}, \tag{47}$$

where it is important to note that in general the quasiparticle energies (21) are all (weakly) dependent upon β. The pressure $\mathcal{P}$ is given by the expression

$$\begin{aligned}\mathcal{P}\Omega&=\langle N\rangle g-\Psi\\&=\xi\langle N\rangle g-\tfrac{1}{2}\xi^2\langle N\rangle W_{0,0}(00|00)-\beta^{-1}\int_0^\beta dtE_B(t)+\beta^{-1}\sum_{\mathbf{p}}\{\ln[1+\nu'(\mathbf{p})]+\tfrac{1}{2}\ln[1+F_-(\mathbf{p})]\},\end{aligned} \tag{48}$$

where it is well to observe that for He II this quantity must vanish.

The fractional occupation of the zero-momentum state ξ, Eq. (10), is given with the aid of Eq. (30) by the expression

$$\xi=1-(n\Omega)^{-1}\sum_{\mathbf{p}}[1-\alpha^2(\mathbf{p})]^{-1}[\alpha^2(\mathbf{p})+\nu'(\mathbf{p})]. \tag{49}$$

Finally, the fractional normal-fluid density δ_N can be calculated by using the expression

$$\delta_N=(n\Omega)^{-1}\sum_{\mathbf{p}}\nu'(\mathbf{p})F_+(\mathbf{p})=(n\Omega)^{-1}\sum_{\mathbf{p}}\nu'(\mathbf{p})[1-\alpha^2(\mathbf{p})]^{-1}. \tag{50}$$

4. PREDICTIONS OF QUANTUM STATISTICS

The purpose of the preceding two sections has been to present a framework, the quasiparticle model, for understanding in a simple way the predictions of the quantum-statistical theory of a Bose fluid.[2] The fundamental interaction quantity which enters into this theory is the function $W_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)$ which reduces to the quantity of Eq. (9) in the weak-interaction limit.

$$W_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)\equiv-\tfrac{1}{2}\langle N\rangle g_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4), \tag{51}$$

where i and j can each be $+$, $-$, 0, or 1 and from Eqs. (6) and (39) of MII we have

$$\begin{aligned}g_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)\equiv{}& f_1(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)+\sum_{\mathbf{k}_5\mathbf{k}_6}f_2(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_5\mathbf{k}_6|\mathbf{k}_3\mathbf{k}_4)\\ &\times\left[P\left(\frac{1}{\epsilon(\mathbf{k}_1)+\epsilon(\mathbf{k}_2)-\epsilon(\mathbf{k}_5)-\epsilon(\mathbf{k}_6)}\right)-P\left(\frac{1}{\epsilon_i(\mathbf{k}_1)+\epsilon_j(\mathbf{k}_2)-\epsilon(\mathbf{k}_5)-\epsilon(\mathbf{k}_6)}\right)\right]\\ ={}&-[\langle\mathbf{k}_3\mathbf{k}_4|G_{ij}|\mathbf{k}_1\mathbf{k}_2\rangle+\langle\mathbf{k}_4\mathbf{k}_3|G_{ij}|\mathbf{k}_1\mathbf{k}_2\rangle].\end{aligned} \tag{52}$$

Explicit expressions for the energies $\epsilon_+(\mathbf{k})$, $\epsilon_-(\mathbf{k})$, and $\epsilon_1(\mathbf{k})$ of Eq. (21) will be given below, and $\epsilon_0(\mathbf{k})\equiv0$. The quantity $\epsilon(\mathbf{k})$ is given by Eq. (11).

The second line of Eq. (52) is a result which has been derived previously by Mohling, and explicit expressions for the functions f_1 and f_2 in terms of free-particle reaction matrices have also been given by him.[12] The operator G_{ij} is a two-particle reaction matrix defined by the operator equation

$$G_{ij}=V+VP[(E_{ij}+2g-H_2)^{-1}]V, \tag{53}$$

where H_2 is the two-particle Hamiltonian (including V) and P denotes that the principal value of the energy denominator is to be taken. The energy E_{ij} is given by $[\epsilon_i(k_1)+\epsilon_j(k_2)]$ in the (real) matrix elements of (52). It must finally be mentioned that the two lines of (52) are not quite correct if one considers the mathematical idealization of an infinite repulsive core as part of the basic two-particle interaction V.

The reaction matrix G_{ij} is not a Hermitean matrix, so that the order of the $\mathbf{k}$ indices in (51) is important. The relation between $g_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)$ and $g_{m,n}(\mathbf{k}_3\mathbf{k}_4|\mathbf{k}_1\mathbf{k}_2)$ is[13]

$$\begin{aligned}g_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)={}&g_{m,n}(\mathbf{k}_3\mathbf{k}_4|\mathbf{k}_1\mathbf{k}_2)+\sum_{\mathbf{k}_5\mathbf{k}_6}f_2(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_5\mathbf{k}_6|\mathbf{k}_3\mathbf{k}_4)\\ &\times\left[P\left(\frac{1}{\epsilon_m(\mathbf{k}_3)+\epsilon_n(\mathbf{k}_6)-\epsilon(\mathbf{k}_5)-\epsilon(\mathbf{k}_6)}\right)-P\left(\frac{1}{\epsilon_i(\mathbf{k}_1)+\epsilon_j(\mathbf{k}_2)-\epsilon(\mathbf{k}_5)-\epsilon(\mathbf{k}_6)}\right)\right],\end{aligned} \tag{54}$$

where f_2 is symmetric under the interchange $(\mathbf{k}_1\mathbf{k}_2)\rightleftarrows(\mathbf{k}_3\mathbf{k}_4)$.

The quantum-statistical theory of a Bose fluid which was developed in MI and MII was applied to a model system, the dilute gas of Bose hard spheres, in MIII. In this latter paper a convention was established for denoting the "order" of any calculated term, which was that the order equals the number of independent momenta in the corresponding graph. According to this convention, the results described in the preceding section are those of a complete first-order calculation. Now, in Sec. 7 of MIII the general first-order expressions for the quasiparticle energies of (21) were derived. Thus, from Eqs. (6), (84), and (86) in MIII we have the results

$$\epsilon_1(\mathbf{p})=\epsilon(\mathbf{p})+2\xi W_{10}(\mathbf{p}0|\mathbf{p}0), \tag{55}$$

$$\begin{aligned}[\epsilon_i(\mathbf{p})+\epsilon_1(-\mathbf{p})]\\ =C_i(\mathbf{p})+i[C_i^2(\mathbf{p})-\xi^2W^2{}_{0,0}(00|\mathbf{p},-\mathbf{p})]^{1/2},\end{aligned} \tag{56}$$

$$C_i(\mathbf{p})=\epsilon(\mathbf{p})+\xi[W_{i,0}(\mathbf{p}0|\mathbf{p}0)+W_{1,0}(-\mathbf{p}0|-\mathbf{p}0)],$$

where $i=+$ or $-$ only in these last expressions. In the weak interaction limit Eqs. (55) and (56) clearly reduce to Eqs. (15) and (20), respectively. By referring to Eqs. (51) and (52) it can be seen that the expressions (55) and (56) are actually rather complicated coupled integral equations for the energies ϵ_1, ϵ_+, and ϵ_-.

The derivation of the first-order expressions for the quantities $f_+(\mathbf{p})$ and $f_-(\mathbf{p})$ is given in Appendix A. The

[12] F. Mohling, Phys. Rev. **124**, 583 (1961); **128**, 1365 (1962).

[13] See Eqs. (30) and (66) of F. Mohling, Phys. Rev. **124**, 583 (1961).

result, Eq. (A6) is

$$f_{\pm}(\mathbf{p})\cong g_{0,0}^{-1}(00|\mathbf{p}-\mathbf{p})[\epsilon_{+}(\mathbf{p})-\epsilon_{-}(\mathbf{p})]^{-1} \times g_{\mp,1}(\mathbf{p}-\mathbf{p}|00)[\epsilon_{\pm}(\mathbf{p})+\epsilon_{1}(-\mathbf{p})]. \quad (57)$$

Similarly, the quantum-statistical derivation of Eqs. (40) and (42) is given in Appendix B. An important consequence of this derivation are the $\mathbf{p}=0$ results given in connection with Eqs. (B3) and (B4).

$$\alpha(0)=\alpha_{+}(0)=\alpha_{-}(0)=1. \quad (58)$$

As a final matter for this section, we consider the quantum-statistical derivation of Eq. (45) for the Helmholtz free energy to first order. With the aid of Eqs. (8), (9), (47), (57)–(60) of MIII and Eqs. (6), (38), (40), (107), (109), (110), and (115) of MII one can show that the first-order expression for the grand potential is

$$f(n,\beta,\Omega)=\tfrac{1}{2}\Omega^{-1}\sum_{\mathbf{p}}\ln[1+\nu'(\mathbf{p})][1+\langle n(\mathbf{p})\rangle] +\xi n\beta[g-\tfrac{1}{2}\xi W_{0,0}(00|00)] -\Omega^{-1}\int_{0}^{\beta}dtE_{B}(t), \quad (59)$$

where $E_B(t)$ is given by Eq. (46). In the derivation of the expression for $E_B(t)$ it is necessary to use Eqs. (A1) of Appendix A and to make approximations of the type (A5). One then arrives at Eq. (45) for the Helmholtz free energy by using Eq. (30) and the expression

$$\Psi=\langle N\rangle g-\beta^{-1}\Omega f. \quad (60)$$

Instead of Eq. (60) one may equivalently use Eq. (13′) of MI together with Eq. (43).

5. CHEMICAL POTENTIAL

The chemical potential g is an important quantity for which we have not yet given an expression. This quantity is required in the theory in such relations as Eq. (45) for the Helmholtz free energy and Eqs. (55) and (56) for the quasiparticle energies. In the latter equations g occurs by virtue of its appearance in Eq. (11). To be sure, it is necessary to find an expression for g, because the useful thermodynamic independent variables for a Bose fluid are n, β, and Ω. As we have remarked above Eq. (43), the chemical potential can be calculated as the self-energy of a zero-momentum boson. In fact, this last statement is the basis of Eqs. (2) and (49) of MIII for the first-order determination of g by the relation

$$g=-[\Delta_0^{(0)}+\Delta_1^{(0)}+\Delta_2^{(0)}]. \quad (61)$$

The explicit quantum-statistical calculation of $\Delta_0^{(0)}$, $\Delta_1^{(0)}$, and $\Delta_2^{(0)}$ is outlined in detail in Sec. 4 of MIII and after tedious manipulations and integrations one obtains the following general results:

$$-\Delta_0^{(0)}=\xi W_{00}(00|00)-\Delta_{01}^{(0)}, \quad (62)$$

$$\begin{aligned}-\Delta_1^{(0)}=&\,2(n\Omega)^{-1}\sum_{\mathbf{p}}P_{+}^{-1}(\mathbf{p})P_{-}^{-1}(\mathbf{p})P(\mathbf{p})[\epsilon_{+}(\mathbf{p})-\epsilon_{-}(\mathbf{p})]^{-1}[P_{-}(\epsilon_{-}+\epsilon_1)W_{-0}(\mathbf{p}0|\mathbf{p}0)-P_{+}(\epsilon_{+}+\epsilon_1)W_{+0}(\mathbf{p}0|\mathbf{p}0)]\\ &+4(n\Omega)^{-1}\sum_{\mathbf{p}}F_{-}(\mathbf{p})P(\mathbf{p})(P_{+}\alpha_{+}+P_{-}\alpha_{-})^{-1}[\alpha_{-}W_{-0}(\mathbf{p}0|\mathbf{p}0)+\alpha_{+}W_{+0}(\mathbf{p}0|\mathbf{p}0)]\\ &+2(n\Omega)^{-1}\sum_{\mathbf{p}}\nu'(\mathbf{p})F_{+}(\mathbf{p})P(\mathbf{p})P_{+}^{-2}P_{-}^{-2}(\alpha_{+}-\alpha_{-})\\ &\qquad\times\{-P_{+}\alpha_{+}W_{+0}(\mathbf{p}0|\mathbf{p}0)[1+P_{-}^{2}(\alpha_{+}-\alpha_{-})^{-2}]P_{-}\alpha_{-}W_{-0}(\mathbf{p}0|\mathbf{p}0)[1+P_{+}^{2}(\alpha_{+}-\alpha_{-})^{-2}]\}\\ &\xrightarrow[\beta\to\infty]{}2\cdot(n\Omega)^{-1}\sum_{\mathbf{p}}(\epsilon_{+}-\epsilon_{-})^{-1}(\epsilon_{-}+\epsilon_1)W_{-0}(\mathbf{p}0|\mathbf{p}0), \end{aligned} \quad (63)$$

$$\begin{aligned}-\Delta_2^{(0)}=&-\xi(n\Omega)^{-1}\sum_{\mathbf{p}}[(P_{+}^{-1}-P_{-}^{-1})P(\epsilon_{+}-\epsilon_{-})^{-1}-\tfrac{1}{2}\epsilon^{-1}(\mathbf{p})]W_{0,0}^{2}(00|\mathbf{p}-\mathbf{p})\\ &-2(n\Omega)^{-1}\sum_{\mathbf{p}}F_{-}(\mathbf{p})\,,P\alpha_{+}\alpha_{-}(P_{+}\alpha_{+}+P_{-}\alpha_{-})^{-1}W_{0,0}(00|\mathbf{p}-\mathbf{p})\\ &-2\xi(n\Omega)^{-1}\sum_{\mathbf{p}}\nu'(\mathbf{p})F_{+}(\mathbf{p})(P_{+}^{-1}-P_{-}^{-1})P(\epsilon_{+}-\epsilon_{-})^{-1}W_{0,0}^{2}(00|\mathbf{p}-\mathbf{p})\\ &\xrightarrow[\beta\to\infty]{}-\xi(n\Omega)^{-1}\sum_{\mathbf{p}}[(\epsilon_{+}-\epsilon_{-})^{-1}-\tfrac{1}{2}\epsilon^{-1}(\mathbf{p})]W_{0,0}^{2}(00|\mathbf{p}-\mathbf{p}). \end{aligned} \quad (64)$$

In these equations, we have used the notation of Eqs. (37) and (38) for $P_{+}(\mathbf{p})$, $P_{-}(\mathbf{p})$, and $P(\mathbf{p})$ [the latter quantity should not be confused with the principal value P which appears in Eqs. (52)–(54)], Eq. (40) for $\alpha_{\pm}(\mathbf{p})$, and Eq. (51) for the interaction energy W_{ij}. The first term in Eq. (62) is the leading contribution to the interaction of a zero-momentum boson with another zero-momentum boson. The first-order contribution to $\Delta_0^{(0)}$ is $\Delta_{01}^{(0)}$, and this quantity is discussed in Appendix C. The first-order quantities $-\Delta_1^{(0)}$ and $-\Delta_2^{(0)}$ are, roughly speaking, perturbation theory contributions to g due to the interaction (51) of a zero-momentum boson with nonzero-momentum bosons, either by "direct interaction" [$\Delta_1^{(0)}$] or by "pair interaction" [$\Delta_2^{(0)}$]. In the expressions for each of these quantities, the first term is due to virtual interactions, the second term is due to real interaction with superfluid bosons, and the third term is due to real interactions with normal-fluid bosons. If one identifies these different terms with their starting points in the derivation (outlined in Sec. 4 of MIII), then one finds that the functions $N_{02}'(\mathbf{p})$ and $N_{20}'(\mathbf{p})$ of the general theory correspond to the superfluid momentum distribution $F_-(\mathbf{p})$, whereas $N_{11}'(\mathbf{p})$ corresponds to the normal-fluid momentum distribution $\nu'(\mathbf{p})F_+(\mathbf{p})$ [see Eqs. (C32)]. In the zero-temperature limit, Eqs. (62)–(64) are in agreement with Eqs. (53)–(55) of MIII when one makes the low-density approximation (see Sec. 2 of MIII) $W_{ij}\cong\frac{1}{2}W=4\pi\hbar^2a_s/M[\neq W$ of Eq. (9)], where a_s is the two-particle scattering length. According to Eqs. (63) and (64), only the virtual interaction terms survive in the zero-temperature limit. This last result is consistent with the general expectation that ground-state expressions for a Bose fluid should be the same as those for a Boltzmann fluid and therefore that they should be independent of statistics, i.e., independent of any momentum distributions. We finally observe that all of our $T=0$ results, e.g., Eqs. (42), (46), (63), and (64), depend only upon superfluid quantities, as they should since there is no normal fluid at $T=0$.

The assumption that g can be calculated as the self-energy of a zero-momentum boson can always be checked because there is one thermodynamic relation Eq. (3′) of MI, which has not yet been used. According to Eq. (60) this relation may be written as

$$(\partial\Psi/\partial\xi)|_{n,\beta,\Omega}=0 \quad (\text{when } \xi\neq 0), \tag{65}$$

that is to say, if the chemical potential has been correctly calculated, then the Helmholtz free energy will be minimal with respect to ξ.[14] Of course, the relation for ξ given by Eq. (49) must not be substituted into Ψ before computing the derivative (65).

Using the fact that $G=\langle N\rangle g=\langle E\rangle-TS+\mathcal{P}\Omega$, one can see for a bound system ($\mathcal{P}=0$) that the chemical potential is always negative and $\geqq$ the binding energy per particle in absolute value. For He II $g\sim-(10°\mathrm{K})\kappa$ and this observation can be used to justify the approximation (A5) of Appendix A.

6. DISCUSSION

The purpose of this paper has been to present the general first-order predictions of the quantum-statistical theory for a Bose fluid within the framework of a quasiparticle model. Several questions for further examination as well as some possible implications of the theory will be discussed in this section.

The first, and most pressing, question concerns the effect of a second-order calculation on our conception of the quasiparticle model, i.e., does the model continue to provide useful insight into the microscopic physics of a Bose fluid when a second-order calculation is performed. This question is particularly significant in the "high-" temperature region $T\lesssim T_\lambda$ where T_λ is the transition temperature for Bose-Einstein condensation. Thus, most of the results of this paper reduce to well-known free Bose-gas results in the limit $\xi\to 0^+$, which is equivalent to the limit $T\to T_\lambda^-$. The description of the transition temperature region is therefore not included in most of the thermodynamic expressions of this paper. Only the chemical potential has a nonvanishing first-order contribution when $\xi\to 0$, which is given by $g=-\Delta_1^{(0)}$ [see Eqs. (61)–(64)].

A second question of great interest is the question of two-particle distribution functions in the Bose fluid. In Sec. 8 of MI, a general prescription for calculating the pair-distribution function was written down, a prescription which can easily be extended to the calculation of the off-diagonal two-particle density matrix elements. Now, on the one hand, the Fourier transform of the pair-distribution function is a quantity [usually called $S(p)$] which can be deduced experimentally from cross-section measurements.[15] On the other hand, the off-diagonal density matrix elements in position space must be characterized by off-diagonal long-range order (ODLRO), a concept originally discussed by Penrose[16] and by Penrose and Onsager[5] and then further clarified by Yang.[17] The study of ODLRO can shed further light on the microscopic understanding of the Bose fluid.

The question of "order" in the Bose fluid also involves studying the mixed-state wave function Φ_T of Eq. (32), in which the basic pair wave function $f(\mathbf{r})$, Eq. (34), is the primary quantity of interest. Referring to expression (42) for the amplitude $\alpha(\mathbf{p})$, we see that $f(\mathbf{r})$ has the property that the normal-fluid and superfluid components have the opposite sign, or phase. This fact suggests that there is an intimate relationship between the wave function Φ_T and the wave functions which Feynman discusses in his atomic theory.[1] A study of this relationship, together with the study of the two-particle distribution function may perhaps also yield greater insight into the role of the Feynman-Bijl relation[1,18] for the excitation energies in a Bose fluid

$$\omega_{\mathrm{exc}}(\mathbf{p})=S^{-1}(\mathbf{p})\omega(\mathbf{p}). \tag{66}$$

This brings us to the final question to be discussed here, which is: What are the excitation energies in a Bose fluid? Of course, Landau's original suggestion[19] that

[14] Equation (65) is a minimal condition rather than a maximal condition, because at constant n, β, and Ω, the free energy is a minimum with respect to any possible changes of state for an equilibrium system. See, e.g., L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, Inc., New York, 1958), Chap. 2.

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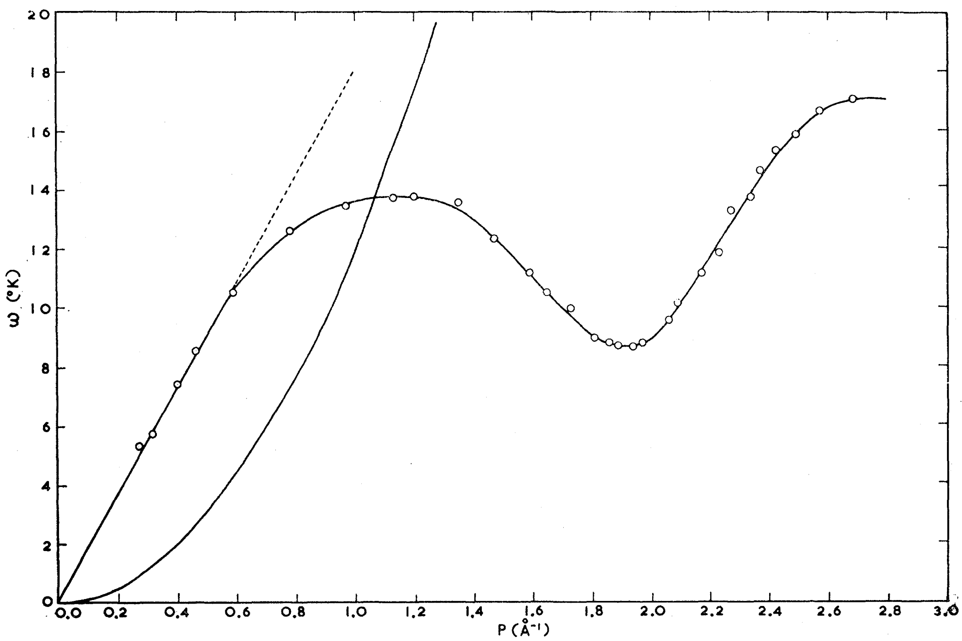

FIG. 1. Excitation spectrum for He II at $T=1.12$°K, as measured by Henshaw and Woods (Ref. 20), with inelastically scattered thermal neutrons of incident wavelength 4.04 Å. The dashed straight line is the phonon curve $\omega=cp$ where $c=237$ m/sec, and the parabola is the (two) free-particle curve $\omega=\hbar^2p^2/M$.

there is a single energy momentum relation $\omega_{\text{exc}}(p)$ seems to have been verified experimentally for He II.[20] Nevertheless, our quasiparticle model suggests that there are two excitation energies $\omega_+(\mathbf{p})$ and $\omega_-(\mathbf{p})$ in a Bose fluid, where

$$\omega_\pm(\mathbf{p})\equiv\pm[\epsilon_\pm(\mathbf{p})+\epsilon_1(-\mathbf{p})-g]\underset{\mathbf{p}\to 0}{\longrightarrow} cp. \tag{67}$$

The motivation for the definition (67) is Eq. (B4) of Appendix B which together with the leading term in the chemical potential, Eqs. (61) and (62), shows that it is the definition (67) which is capable of yielding a linear, or phonon, momentum dependence in the limit $p\to 0$. Therefore, we are led to suggest that the elementary excitations in a Bose fluid are:

(1) The destruction of a superfluid (hole) quasiparticle, corresponding to the energy $\omega_-(\mathbf{p})\to -g$ in the limit $\mathbf{p}\to\infty$.

(2) The creation of a normal-fluid quasiparticle, corresponding to the energy $\omega_+(\mathbf{p})\to 2\omega(\mathbf{p})-3g$ in the limit $p\to\infty$.

In Fig. 1 we reproduce an experimental excitation curve for He II as measured by Henshaw and Woods.[20] The behavior of this curve relative to the curve for $2\omega_\mathrm{p}$ also shown, and its turning down in the region $\rho\gtrsim 2.5$ Å^{-1}, strongly suggests that Henshaw and Woods have measured the curve $\omega_-(\mathbf{p})$. Our quasiparticle model indicates that the curve for $\omega_+(\mathbf{p})$ probably undergoes a smooth transition from the straight-line phonon behavior $\omega=cp$ to the parabola $[2\omega(\mathbf{p})-3g]$ in the region $(0.6\text{ Å}^{-1})\leq p\leq(1.2\text{ Å}^{-1})$ (assuming that both quasiparticles are characterized by the same phonon velocity c). In the lower part of this interval the double-quasiparticle spectrum should be observable, whereas for higher momenta the $\omega_+(\mathbf{p})$ excitations may be quite unstable.

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APPENDIX A

The quantum-statistical derivation of the quantities $f_+(\mathbf{p})$ and $f_-(\mathbf{p})$ to first order, is achieved by referring to Eqs. (21) and (81) of MIII and (47) of MII. Thus, one finds for the basic functions $A_i^{(<)}$ and $A_i^{(>)}$ of the Λ transformation

$$A_\pm^{(<)}(t,\mathbf{k})=D_1^{-2}(\beta,t,\mathbf{k})\zeta(\beta,\mathbf{k})e^{\beta\epsilon_\mp(\mathbf{k})}\{G_-(\beta,\mathbf{k})[\epsilon_+(\mathbf{k})+\epsilon_1(-\mathbf{k})]-G_+(\beta,\mathbf{k})[\epsilon_-(\mathbf{k})+\epsilon_1(-\mathbf{k})]\}^{-1}$$
$$\times\{G_-(t,\mathbf{k})[\epsilon_+(\mathbf{k})+\epsilon_1(-\mathbf{k})]f_-(\mathbf{k})e^{-t\epsilon_+(\mathbf{k})}-G_+(t,\mathbf{k})[\epsilon_-(\mathbf{k})+\epsilon_1(-\mathbf{k})]f_+(\mathbf{k})e^{-t\epsilon_-(\mathbf{k})}\}, \tag{A1}$$
$$A_\pm^{(>)}(t,\mathbf{k})=D_1^{-1}(\beta,t,\mathbf{k})\zeta(t,\mathbf{k})f_\pm(\mathbf{k}),$$

to first order, where $G_i(t,\mathbf{k})$ is given by Eq. (82) in MIII and

$$\zeta(t,\mathbf{k})=D_1(t,t,\mathbf{k})[f_+(\mathbf{k})e^{\beta\epsilon_+(\mathbf{k})}-f_-e^{\beta\epsilon_-(\mathbf{k})}]^{-1}, \tag{A2}$$

[20] D. G. Henshaw and A. D. B. Woods, Phys. Rev. 121, 1266 (1961). References to previous experiments are given in this article.

$$D_1(t_2,t_1,\mathbf{k})\equiv\{G_-(\beta,\mathbf{k})[\epsilon_+(\mathbf{k})+\epsilon_1(-\mathbf{k})]-G_+(\beta,\mathbf{k})[\epsilon_-(\mathbf{k})+\epsilon_1(-\mathbf{k})]\}^{-1}$$
$$\times\{G_-(t_1,\mathbf{k})[\epsilon_+(\mathbf{k})+\epsilon_1(-\mathbf{k})]e^{(\beta-t_2)\epsilon_+(\mathbf{k})}-G_+(t_1,\mathbf{k})[\epsilon_-(\mathbf{k})+\epsilon_1(-\mathbf{k})]e^{(\beta-t_2)\epsilon_-(\mathbf{k})}\}\xrightarrow[t_2=t_1=\beta]{}1\,, \quad \text{(A3)}$$

$$f_\pm(\mathbf{k})\equiv\{G_-(\beta,\mathbf{k})[\epsilon_+(\mathbf{k})+\epsilon_1(-\mathbf{k})]-G_+(\beta,\mathbf{k})[\epsilon_-(\mathbf{k})+\epsilon_1(-\mathbf{k})]\}^{-1}g_{\mp,1}(\mathbf{k}-\mathbf{k}|00)[\epsilon_\pm(\mathbf{k})+\epsilon_1(-\mathbf{k})]. \quad \text{(A4)}$$

Upon comparing Eqs. (92) and (115) of MII with Eqs. (39), (A2), and (A3) of the present paper, one sees that the functions defined by Eq. (A4) must be identical with those of Eqs. (36).

Now to good approximation, one may write (see end of Sec. 5)

$$G_i(t,\mathbf{k})\cong g_{00}(00|\mathbf{p}-\mathbf{p})\,, \quad \text{(A5)}$$

and in this case Eqs. (A1)–(A4) are greatly simplified. For example, the expression for $D_1(\beta,t,\mathbf{k})$ becomes unity in this approximation. Similarly, the expression for $f_\pm(\mathbf{k})$ becomes

$$f_\pm(\mathbf{k})\cong g_{0,0}^{-1}(00|\mathbf{k}-\mathbf{k})[\epsilon_+(\mathbf{k})-\epsilon_-(\mathbf{k})]^{-1}$$
$$\times g_{\mp,1}(\mathbf{k}-\mathbf{k}|00)[\epsilon_\pm(\mathbf{k})+\epsilon_1(-\mathbf{k})], \quad \text{(A6)}$$

when the approximation (A5) is made.

APPENDIX B

In this appendix we give the quantum-statistical derivation of Eq. (42) for $\alpha(p)$. The derivation begins by referring to the first-order expression for the momentum distribution, Eq. (73) of MIII, which upon comparison with Eq. (30) of the present paper yields for $\alpha(p)$ the general relation

$$\alpha(\mathbf{p})=-\nu'(\mathbf{p})\zeta^{-1}(\beta,\mathbf{p})K_{0,2}'(\mathbf{p})$$
$$=-\nu'(\mathbf{p})e^{-\beta\epsilon_1(-\mathbf{p})}K_{2,0}'(\mathbf{p})\,. \quad \text{(B1)}$$

We have used Eqs. (92), (97), and (115) of MII in order to obtain the particular forms of (B1), and the minus sign has been introduced in order that agreement with Eqs. (40) and (42) can be finally achieved.

If one substitutes Eqs. (5) and the first line of (32) from MIII, Eq. (38) from MII, and Eqs. (A1) of the present paper into the second line of (B1), then one obtains Eq. (42) directly if the identifications

$$\alpha_\pm(\mathbf{p})\equiv-\tfrac{1}{2}\langle N\rangle\xi G_\mp(\beta,\mathbf{p})[\epsilon_\mp(\mathbf{p})+\epsilon_1(-\mathbf{p})]^{-1} \quad \text{(B2)}$$

are made. The approximation (A5) together with Eq. (51) then yields Eq. (40).

(0,2) Difficulty

When the first-order calculation of $\alpha(\mathbf{p})$ is made using the first line of Eq. (B1), then one does not obtain Eq. (B2) directly. That is to say, the "complete" first-order calculation of $K_{0,2}'(\mathbf{p})$ differs from that of $K_{2,0}'(\mathbf{p})$ by more than simple multiplying factors. Upon using the identity (54) one can show that the difference between the two calculations involves higher powers of the function f_2 [see Eq. (52) for notation] which may be neglected to first order. It is in this sense that the two lines of (B1) yield the same result.

It seems whenever (0,2) functions are calculated that higher order differences with the corresponding (2,0) functions occur. Thus, the identity between the two lines of (B1), which is quite general, does not hold exactly to specific orders when graphical iterations are performed. Of course, this (0,2) difficulty is not an inconsistency of the theory, if only higher order calculations cancel the differences which occur at lower orders. This is an aspect of the theory which we have not yet studied in great detail.

There is an another aspect to the (0,2) difficulty, which is: How can we say that it is a (0,2) difficulty instead of a (2,0) difficulty? That is, which line of (B1) gives the "correct," or best, first-order calculation of $\alpha(\mathbf{p})$. This question is answered by referring to the general identities (102) of MII.

$$\lim_{\mathbf{p}\to 0}[\nu'(\mathbf{p})]^{-1}=\lim_{\mathbf{p}\to 0}[-\zeta^{-1}(\beta,\mathbf{p})K_{0,2}'(\mathbf{p})]$$
$$=\lim_{\mathbf{p}\to 0}[-e^{-\beta\epsilon_1(-\mathbf{p})}K_{2,0}'(\mathbf{p})], \quad \text{(B3)}$$

where we have used Eq. (115) of MII and Eqs. (6) and (30) of MIII. According to Eq. (B1) these last identities are equivalent to the identity $\alpha(0)\equiv 1$. The second line of (B3) can be satisfied if one sets

$$\lim_{\mathbf{p}\to 0}[\epsilon_i(\mathbf{p})+\epsilon_1(-\mathbf{p})]=\lim_{\mathbf{p}\to 0}[-\tfrac{1}{2}\langle N\rangle\xi G_i(\beta,\mathbf{p})]$$
$$\cong\xi W_{0,0}(00|00)\quad(i=\pm)\,, \quad \text{(B4)}$$

where we have again made the approximation (A5) to obtain the second line of (B4). Thus, according to Eq. (B2), Eq. (B4) is equivalent to the identities $\alpha_\pm(0)\equiv 1$. If, alternatively, we calculate the right-hand side of the first line of (B3) by using the "complete" first-order expression for $K_{0,2}'(\mathbf{p})$, then it becomes extremely difficult to see how the identities (B3) can be satisfied to "first" order. It is this fact which justifies our emphasis on the (0,2) difficulty rather than on a (2,0) difficulty. The occurrence of the (0,2) difficulty is due to the different ways in which (0,2) and (2,0) functions are Λ transformed [see, e.g., Eqs. (76), and (77) in MII]. The (0,2) difficulty also occurs in the derivation of Eqs. (63) and (64) for $\Delta_1^{(0)}$ and $\Delta_2^{(0)}$.

APPENDIX C

In this Appendix we discuss the derivation of the quantity $\Delta_{01}^{(0)}$ of Eq. (62) which is the first-order contribution to the interaction of a zero-momentum boson with itself. The derivation follows the pattern presented in Sec. 4 of MIII (especially steps 2 and 8), except for the following modification (1).

(1) Unlike the approximation mentioned in step 6 of Sec. 4 of MIII, we use the complete expression for the pair function as given by Eq. (38) of MII to obtain

$$\mathcal{K}_{\text{out}}'(t)_0=\int_0^\beta ds G_{\text{out}}'(s)\mathcal{K}'(s,t)\,,$$
$$\mathcal{K}_{\text{in}}'(t_0)=\int_0^\beta ds\mathcal{K}'(t,s)G_{\text{in}}'(s)\,, \tag{C1}$$

where

$$\mathcal{K}'(t,s)\equiv\tfrac{1}{2}x\Omega\int_0^\beta dt' G_{\text{out}}'(t')^{tt'}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}'_s G_{\text{in}}'(s)\,. \tag{C2}$$

Then, according to Eqs. (109), (110), and (112) of MII

$$[\Delta_{00}{}^{(0)}+\Delta_{01}{}^{(0)}]\theta(t-s)=t\text{- and }s\text{-independent part of }\mathcal{K}'(t,s)\,. \tag{C3}$$

With $\Delta_{00}{}^{(0)}=-\xi W_{00}(00|00)$, we obtain:

$$\Delta_{01}{}^{(0)}\theta(t-s)=t\text{- and }s\text{-independent part of }\mathcal{K}''(t,s)\,, \tag{C4}$$

where

$$\mathcal{K}''(t,s)\equiv\tfrac{1}{2}x\Omega\int_t^\beta dt'[\delta(\beta-t')K_{\text{in}}'(s)+K_{\text{out}}'(t')+K_{\text{out}}'(t')K_{\text{in}}'(s)]^{tt'}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}'_s$$
$$\cong\tfrac{1}{2}x\Omega\int_t^\beta dt'[\delta(\beta-t')K_{\text{in}}'(s)+K_{\text{out}}'(t')]^{tt'}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}'_s \tag{C5}$$

and we have neglected the product term in the second line of (C5) since it gives a higher order contribution.

(2) We next write the quantities $K_{\text{in}}'(t)$ and $K_{\text{out}}'(t)$ as expansions corresponding to their diagrammatic origins. [See Eqs. (50) and (51) of MIII.]

$$K_{\text{in}}'(t)=K_{\text{in}}'(t)_0+K_{\text{in}}'(t)_1+K_{\text{in}}'(t)_2+\cdots\,,$$
$$K_{\text{out}}'(t)=K_{\text{out}}'(t)_0+K_{\text{out}}'(t)_1+K_{\text{out}}'(t)_2+\cdots\,, \tag{C6}$$

where the first term in each of these two expansions is given, respectively, by the equations

$$K_{\text{in}}'(t)_0=\int_0^\beta ds K''(t,s)G_{\text{in}}'(s)\,,$$
$$K_{\text{out}}'(t)_0=\int_0^\beta ds G_{\text{out}}'(s)K''(s,t) \tag{C7}$$

with

$$K''(t,s)\equiv\mathcal{K}''(t,s)-\Delta_{01}{}^{(0)}\theta(t-s)\,. \tag{C8}$$

The term

$$-\textstyle\sum_{\mathbf{k}} K_0{}^{(1)}(t,s,\mathbf{k})\,,$$

where

$$K_0{}^{(1)}(t,s,\mathbf{k})=\tfrac{1}{2}x\Omega\{\exp-(t-s)[\epsilon(\mathbf{k})+\epsilon(-\mathbf{k})]\}f_2(00|\mathbf{k}-\mathbf{k}|00)P\left(\frac{1}{\epsilon(\mathbf{k})+\epsilon(-\mathbf{k})}\right), \tag{C9}$$

which should really be included on the right-hand side of (C8) is more conveniently treated by adding it to the function $-\sum_{\mathbf{k}} K_2{}^{(1)}(t,s,\mathbf{k})$ of Eqs. (C30) and (C31) (in this connection, see step 5 in Sec. 4 of MIII). We observe that Eqs. (C7) are actually coupled integral equation for the determination of $K_{\text{in}}'(t)_0$ and $K_{\text{out}}'(t)_0$, as one can see explicitly by referring to Eqs. (C4)–(C6) and (C8). To the order which we are calculating one may replace $G_{\text{in}}'(s)$ by 1 and $G_{\text{out}}'(s)$ by $\delta(\beta-s)$ in Eqs. (C8). Equations (C6) together with the first-order solutions for $K_{\text{in}}'(t)_0$ and $K_{\text{out}}'(t)_0$ must be substituted into (C5) to determine $\Delta_{01}{}^{(0)}$ by (C4).

(3) We may write $\Delta_{01}{}^{(0)}$ as a sum of three terms

$$\Delta_{01}{}^{(0)}=\Delta_{0a}{}^{(0)}+\Delta_{0b}{}^{(0)}+\Delta_{0c}{}^{(0)}\,, \tag{C10}$$

where [see Eq. (C4)] $\Delta_{0a}{}^{(0)}$ comes from $\mathcal{K}''(t,s)_1$, $\Delta_{0b}{}^{(0)}$ comes from $\mathcal{K}''(t,s)_2$, and $\Delta_{0c}{}^{(0)}$ comes from $\mathcal{K}''(t,s)_0$. Here we refer to the second line of Eq. (C5) to define

$$\mathcal{K}''(t,s)_i\equiv\tfrac{1}{2}x\Omega\int_t^\beta dt'[\delta(\beta-t')K_{\text{in}}'(s)_i+K_{\text{out}}'(t')_i]^{tt'}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}_s. \tag{C11}$$

The quantity $K''(t,s)$ of Eq. (C8) can then be written as

$$K''(t,s)\cong K''(t,s)_0+K''(t,s)_1+K''(t,s)_2 \tag{C12}$$

to the order which we calculate, where

$$\begin{aligned} K''(t,s)_0&\equiv \mathcal{K}''(t,s)_0-\Delta_{0c}{}^{(0)}\theta(t-s)\,,\\ K''(t,s)_1&\equiv \mathcal{K}''(t,s)_1-\Delta_{0a}{}^{(0)}\theta(t-s)\,,\\ K''(t,s)_2&\equiv \mathcal{K}''(t,s)_2-\Delta_{0b}{}^{(0)}\theta(t-s)\,. \end{aligned} \tag{C13}$$

We may then solve Eqs. (C7) and (C8) by iteration using in first approximation

$$K''(t,s)\cong K''(t,s)_1+K''(t,s)_2\,. \tag{C14}$$

It is to be emphasized that the consistent first-order calculation of $\Delta_{01}{}^{(0)}$, in which terms to $O(f_2)$, say, are retained and higher order terms, e.g., $O(f_2{}^2)$ terms, are neglected, requires the approximate calculation of $K''(t,s)_0$, via Eqs. (C11) and (C13). The function $K''(t,s)_0$ must then be substituted into Eqs. (C7) in order to obtain a better solution for $K_{\rm in}'(t)_0$ and $K_{\rm out}'(t)_0$, etc. Fortunately, this iteration procedure ends after a few iterations, with only $O(f_2{}^2)$ or smaller terms remaining in $K''(t,s)_0$. Thus, it should be clear that $\Delta_{01}{}^{(0)}$ can be determined from the preceding equations as soon as we know $K_{\rm out}'(t)_{1,2}$ and $K_{\rm in}'(t)_{1,2}$. The remainder of the Appendix will therefore be devoted to the expressions for these latter quantities.

(4) In order to proceed further it is useful to define the following quantities which appear in the expressions for $K_{\rm out}'(t)_1$, $K_{\rm out}'(t)_2$, $K_{\rm in}'(t)_1$, and $K_{\rm in}'(t)_2$.

$$J_1(\mathbf{p})=1+\nu'(\mathbf{p})F_+(\mathbf{p})\{1-P_+{}^{-1}(\mathbf{p})P_-{}^{-1}(\mathbf{p})[\alpha_+(\mathbf{p})-\alpha_-(\mathbf{p})]^2\}\,, \tag{C15}$$

$$J(\mathbf{p})=J_1(\mathbf{p})-\nu'(\mathbf{p})F_-(\mathbf{p})\alpha^{-1}(\mathbf{p})[\alpha_+(\mathbf{p})-\alpha_-(\mathbf{p})][P_+{}^{-1}(\mathbf{p})-P_-{}^{-1}(\mathbf{p})]\,, \tag{C16}$$

$$H_\pm{}^{(1)}(\beta,\mathbf{p})=P_\pm{}^{-1}(\mathbf{p})P(\mathbf{p})\{\exp\beta[\epsilon_\mp(\mathbf{p})-\epsilon_\pm(\mathbf{p})]\}[\epsilon_\pm(\mathbf{p})-\epsilon_\mp(\mathbf{p})]^{-1}[\epsilon_\mp(\mathbf{p})+\epsilon_1(-\mathbf{p})]J(\mathbf{p})\,, \tag{C17}$$

$$\begin{aligned} H_\pm{}^{(2)}(\beta,\mathbf{p})=\Delta P(\mathbf{p})P_\mp{}^{-1}(\mathbf{p})&[\epsilon_\mp(\mathbf{p})-\epsilon_\pm(\mathbf{p})]^{-1}[\epsilon_\pm(\mathbf{p})+\epsilon_1(-\mathbf{p})]J_1(\mathbf{p})\\ &+[\epsilon_\pm(\mathbf{p})+\epsilon_1(-\mathbf{p})][\epsilon_\pm(\mathbf{p})-\epsilon_\mp(\mathbf{p})]^{-1}\nu'(\mathbf{p})F_+(\mathbf{p})\\ &\qquad+2\nu'(\mathbf{p})F_-(\mathbf{p})P(\mathbf{p})\alpha^{-1}(\mathbf{p})P_+{}^{-1}(\mathbf{p})P_-{}^{-1}(\mathbf{p})\alpha_\pm(\mathbf{p})\,, \end{aligned} \tag{C18}$$

$$B_{ij}=\epsilon_i(\mathbf{k}_1)+\epsilon_j(\mathbf{k}_2)-\epsilon(\mathbf{k}_5)-\epsilon(\mathbf{k}_6)\,, \tag{C19}$$

$$G_{ij}{}^A(t,\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)=g_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)-\sum_{\mathbf{k}_5\mathbf{k}_6}f_2(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_5\mathbf{k}_6|\mathbf{k}_3\mathbf{k}_4)e^{tB_{ij}}[(B_{ij}-A)^{-1}-B_{ij}{}^{-1}]\,, \tag{C20}$$

$$\begin{aligned} g_{ij}{}^A(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)&=G_{ij}{}^A(0,\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)\\ &=f_1(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)+\sum_{\mathbf{k}_5\mathbf{k}_6}f_2(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_5\mathbf{k}_6|\mathbf{k}_3\mathbf{k}_4)\left[\left(\frac{1}{\epsilon(\mathbf{k}_1)+\epsilon(\mathbf{k}_2)-\epsilon(\mathbf{k}_5)-\epsilon(\mathbf{k}_6)}\right)-\left(\frac{1}{B_{ij}-A}\right)\right], \end{aligned} \tag{C21}$$

where A is some energy function. It is interesting to note the similarity between Eq. (C21) and Eq. (52) for $g_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)$. Special cases of (C20) and (C21) are

$$\begin{aligned} G_{00}{}^{-A_{i1}}(t,00|\mathbf{p}-\mathbf{p})&=G_i(t,\mathbf{p})\,,\\ g_{i1}{}^{A_{i1}}(\mathbf{p}-\mathbf{p}|00)&=g_{00}(00|\mathbf{p}-\mathbf{p})\,,\\ A_{i1}&=\epsilon_i(\mathbf{p})+\epsilon_1(-\mathbf{p})\,, \end{aligned} \tag{C22}$$

where $G_i(t,\mathbf{p})$ was originally defined by Eq. (82) of MIII. Further, more complicated, functions which are encountered are:

$$K_1(t,s,\mathbf{p})=\sum_{i,j=+,-}ijH_i{}^{(1)}(\beta,\mathbf{p})\,{}^t_{(i,0)}\!\begin{bmatrix}\mathbf{p} & 0\\ \mathbf{p} & 0\end{bmatrix}'_s e^{-s\epsilon_j(\mathbf{p})}+\sum_{i=+,-}H_i{}^{(2)}(\beta,\mathbf{p})\sum_{\mathbf{k}_5\mathbf{k}_6}[e^{(t-s)B_{i0}}]B_{i0}{}^{-1}f_2(\mathbf{p}0|\mathbf{k}_5\mathbf{k}_6|\mathbf{p}0)\,, \tag{C23}$$

where $B_{i0}=\epsilon_i(\mathbf{p})-\epsilon(\mathbf{k}_5)-\epsilon(\mathbf{k}_6)$;

$$\begin{aligned} K_2(t,s,\mathbf{p})=\tfrac{1}{4}x\Omega[\epsilon_+(\mathbf{p})-\epsilon_-(\mathbf{p})]^{-1}P(\mathbf{p})&\sum_{i,j=+,-}ije^{(sA_{i1}-tA_{j1})}f_i(\mathbf{p})e^{\beta\epsilon_j(\mathbf{p})}\\ \times\{(1-\delta_{ij})&e^{(t-s)A_{j1}}g_{i1}{}^{A_{j1}}(\mathbf{p}-\mathbf{p}|00)G_i(t-s,\,\mathbf{p})-g_{00}(00|\mathbf{p}-\mathbf{p})G_{i1}{}^{A_{j1}}(t-s,\,\mathbf{p}-\mathbf{p}|00)\\ -\sum_{\mathbf{k}}&f_2(00|\mathbf{k}-\mathbf{k}|\mathbf{p}-\mathbf{p})A_{j1}B_{j1}{}^{-1}(\mathbf{k})[\epsilon(\mathbf{k})+\epsilon(-\mathbf{k})]^{-1}[\delta_{ij}g_{00}(00|\mathbf{p}-\mathbf{p})+\sum_{\mathbf{k}_0}f_2(00|\mathbf{k}_0-\mathbf{k}_0|\mathbf{p}-\mathbf{p})\\ &\times([B_{ij}(\mathbf{k}_0)-A_{j1}]^{-1}-[B_{i1}(\mathbf{k}_0)-B_{j1}(\mathbf{k})]^{-1})(e^{-(s-t)B_{j1}(\mathbf{k})}-e^{-(s-t)B_{i1}(\mathbf{k}_0)})]\}\,, \end{aligned} \tag{C24}$$

where $B_{i1}(\mathbf{k})=A_{i1}-\epsilon(\mathbf{k})-\epsilon(-\mathbf{k})$ and $A_{i1}=\epsilon_i(\mathbf{p})+\epsilon_1(-\mathbf{p})$;

$$
\begin{aligned}
K_2^{(1)}(t,s,\mathbf{p})=K_0^{(1)}(t,s,\mathbf{p})
&-\tfrac{1}{4}x\Omega[\exp(s-t)(\epsilon_{\mathbf{p}}+\epsilon_{-\mathbf{p}})]\{(\epsilon_{\mathbf{p}}+\epsilon_{-\mathbf{p}})^{-1}g_{00}(00|\mathbf{p}-\mathbf{p})f_1(\mathbf{p}-\mathbf{p}|00)\\
&+\sum_{\mathbf{k}} f_2(00|\mathbf{k}-\mathbf{k}|\mathbf{p}-\mathbf{p})(\epsilon_{\mathbf{k}}+\epsilon_{-\mathbf{k}})^{-1}(\epsilon_{\mathbf{k}}+\epsilon_{-\mathbf{k}}-\epsilon_{\mathbf{p}}-\epsilon_{-\mathbf{p}})^{-1}\\
&\times([\exp(s-t)(\epsilon_{\mathbf{k}}+\epsilon_{-\mathbf{k}}-\epsilon_{\mathbf{p}}-\epsilon_{-\mathbf{p}})][g_{00}(00|\mathbf{p}-\mathbf{p})-g_{01}{}^{B_{01}}(\mathbf{p}-\mathbf{p}|00)]\\
&+[f_1(\mathbf{p}-\mathbf{p}|00)+\sum_{\mathbf{k}_0} f_2(00|\mathbf{k}_0-\mathbf{k}_0|\mathbf{p}-\mathbf{p})(\epsilon_{\mathbf{k}}+\epsilon_{-\mathbf{k}}-\epsilon_{\mathbf{p}}-\epsilon_{-\mathbf{p}})\\
&\qquad\times[\exp(s-t)(\epsilon_{\mathbf{k}_0}+\epsilon_{-\mathbf{k}_0}-\epsilon_{\mathbf{p}}-\epsilon_{-\mathbf{p}})](\epsilon_{\mathbf{k}}+\epsilon_{-\mathbf{k}}-\epsilon_{\mathbf{k}_0}-\epsilon_{-\mathbf{k}_0})^{-1}(\epsilon_{\mathbf{p}}+\epsilon_{-\mathbf{p}}-\epsilon_{\mathbf{k}_0}-\epsilon_{-\mathbf{k}_0})^{-1}])\},
\end{aligned}
\tag{C25}
$$

where $B_{01}=\epsilon_1(-\mathbf{p})-\epsilon(\mathbf{k})-\epsilon(-\mathbf{k})$ and $K_0^{(1)}(t,s,\mathbf{p})$ is defined by Eq. (C9).

$$
\begin{aligned}
K_3(t,s,\mathbf{p})=&\tfrac{1}{8}(x\Omega)^2g_{00}{}^3(00|\mathbf{p}-\mathbf{p})[\epsilon_+(\mathbf{p})+\epsilon_1(-\mathbf{p})]^{-1}[\epsilon_-(\mathbf{p})+\epsilon_1(-\mathbf{p})]^{-1}P^2(\mathbf{p})\\
&\times e^{\beta[\epsilon_+(\mathbf{p})+\epsilon_-(\mathbf{p})]}\sum_{i,j=+,-} ijf_i(\mathbf{p})e^{s[\epsilon_i(\mathbf{p})+\epsilon_1(-\mathbf{p})]}(1-\delta_{ij})\\
&\times\{f_i(\mathbf{p})(e^{-s[\epsilon_j(\mathbf{p})+\epsilon_1(-\mathbf{p})]}-e^{-t[\epsilon_j(\mathbf{p})+\epsilon_1(-\mathbf{p})]})+f_j(\mathbf{p})e^{-t[\epsilon_i(\mathbf{p})+\epsilon_1(-\mathbf{p})]}\}\\
&-\tfrac{1}{2}P^2(\mathbf{p})[\epsilon_+(\mathbf{p})-\epsilon_-(\mathbf{p})]^{-2}\{g_{00}(00|\mathbf{p}-\mathbf{p})-\sum_{\mathbf{k}} f_2(00|\mathbf{k}-\mathbf{k}|\mathbf{p}-\mathbf{p})\\
&\times(\exp(s-t)[\epsilon(\mathbf{k})+\epsilon(-\mathbf{k})])[\epsilon(\mathbf{k})+\epsilon(-\mathbf{k})]^{-1}\}\\
&\times\sum_{i=+,-}\{[\epsilon_i(\mathbf{p})+\epsilon_1(-\mathbf{p})]^2e^{2(\beta-s)\epsilon_i(\mathbf{p})}-[\epsilon_+(\mathbf{p})+\epsilon_1(-\mathbf{p})][\epsilon_-(\mathbf{p})+\epsilon_1(-\mathbf{p})]\\
&\qquad\times(1-\delta_{\epsilon_+,-\epsilon_-})e^{(\beta-s)[\epsilon_+(\mathbf{p})+\epsilon_-(\mathbf{p})]}\},
\end{aligned}
\tag{C26}
$$

$$
\begin{aligned}
K_4(\beta,t,\mathbf{p})=&\tfrac{1}{8}(x\Omega)^2P^2(\mathbf{p})f_+(\mathbf{p})f_-(\mathbf{p})e^{\beta[\epsilon_+(\mathbf{p})+\epsilon_-(\mathbf{p})]}g_{00}{}^2(00|\mathbf{p}-\mathbf{p})\\
&\times[\epsilon_+(\mathbf{p})+\epsilon_1(-\mathbf{p})]^{-1}[\epsilon_-(\mathbf{p})+\epsilon_1(-\mathbf{p})]^{-1}\sum_{i\neq j} ije^{(\beta-t)[\epsilon_j(\mathbf{p})-\epsilon_i(\mathbf{p})]}g_{i1}{}^{A_{j1}}(\mathbf{p}-\mathbf{p}|00)\\
&-\tfrac{1}{2}P^2(\mathbf{p})\sum_{i,j=+,-} ij(1-(1-\delta_{ij})\times\delta_{\epsilon_+,-\epsilon_-})f_i(\mathbf{p})f_j(-\mathbf{p})e^{t[\epsilon_i(\mathbf{p})+\epsilon_j(-\mathbf{p})]}\\
&\qquad\times\{g_{ij}(\mathbf{p}-\mathbf{p}|00)+\sum_{\mathbf{k}} f_2(00|\mathbf{k}-\mathbf{k}|\mathbf{p}-\mathbf{p})[e^{(\beta-t)B_{ij}}]B_{ij}{}^{-1}\},
\end{aligned}
\tag{C27}
$$

where

$$B_{ij}=\epsilon_i(\mathbf{p})+\epsilon_j(-\mathbf{p})-\epsilon(\mathbf{k})-\epsilon(-\mathbf{k})$$

and

$$A_{j1}=\epsilon_j(\mathbf{p})+\epsilon_1(-\mathbf{p}).$$

The principal value is understood for energy denominators in the above expressions, and the appearance of a factor $[1-\delta_{\epsilon_+,-\epsilon_-}]$ in Eqs. (C26) and (C27) is discussed in step 7 below.

In the dilute hard-sphere Bose gas (DHSBG) limit of Sec. 4 in MIII the Eqs. (C15)–(C18) become (at $T=0$)

$$J_1(\mathbf{p})=1;\tag{C15a}$$

$$J(\mathbf{p})=0,\tag{C16a}$$

$$H_\pm{}^{(1)}(\beta,\mathbf{p})=0;\tag{C17a}$$

$$H_+{}^{(2)}(\beta,\mathbf{p})=0,\tag{C18a}$$

$$H_-{}^{(2)}(\beta,\mathbf{p})=f_-(\mathbf{p})=[\epsilon_+(\mathbf{p})-\epsilon_-(\mathbf{p})]^{-1}[\epsilon_-(\mathbf{p})+\epsilon_1(-\mathbf{p})].\tag{C18b}$$

The second and third of these last relations, i.e., (C16a) and (C17a), are valid for all T.

(5) Using the definitions given in step 4 above one finds the following results after some tedious manipulations.

$$K_{\text{in}}'(t)_1=\sum_{\mathbf{p}}\int_0^t dsK_1(t,s,\mathbf{p}),\tag{C28}$$

$$K_{\text{out}}'(t)_1=\sum_{\mathbf{p}} K_1(\beta,t,\mathbf{p}),\tag{C29}$$

$$K_{\text{in}}'(t)_2=\sum_{\mathbf{p}}\int_0^t ds\{[1+\nu'(\mathbf{p})F_+(\mathbf{p})]K_2(t,s,\mathbf{p})-K_2{}^{(1)}(t,s,\mathbf{p})+\nu'(\mathbf{p})F_-(\mathbf{p})\alpha^{-1}(\mathbf{p})P^{-1}(\mathbf{p})K_3(t,s,\mathbf{p})\},\tag{C30}$$

$$K_{\text{out}}'(t)_2=\sum_{\mathbf{p}}\{[1+\nu'(\mathbf{p})F_+(\mathbf{p})]K_2(\beta,t,\mathbf{p})-K_2{}^{(1)}(\beta,t,\mathbf{p})+\nu'(\mathbf{p})F_-(\mathbf{p})\alpha^{-1}(\mathbf{p})P^{-1}(\mathbf{p})K_4(\beta,t,\mathbf{p})\},\tag{C31}$$

where the $N_{\mu\nu}'$ functions of the general theory (Sec. 6 of MII or Sec. 3 of MIII) can be identified in these expressions by the following relations

$$\begin{aligned} N_{11}'(\mathbf{p}) &= \nu'(\mathbf{p})F_+(\mathbf{p}), \\ N_{20}'(\mathbf{p}) &= -\nu'(\mathbf{p})F_-(\mathbf{p})P^{-1}(\mathbf{p})\alpha^{-1}(\mathbf{p}), \end{aligned} \tag{C32}$$

and where $\alpha(\mathbf{p})$ is defined by Eq. (42) and $P(\mathbf{p})$ by Eq. (A2). Then, as we have remarked below Eq. (64) $N_{11}'(\mathbf{p})$ may be identified as the normal fluid momentum distribution $\nu'F_+$, and $N_{20}'(\mathbf{p})$ (essentially) as the superfluid momentum distribution F_-.

In the DHSBG limit we find that Eqs. (C28)–(C31) go over to the corresponding expressions which may be derived directly according to the procedure outlined in Sec. 4 of MIII. It is of great value to derive the results (C28)–(C31) in the DHSBG limit first, in order that one may have some check on the correctness of the general expressions when they are subsequently derived. We see from the above expressions (C15)–(C31) that the use of the complete pair functions [Eq. (38), MII] has led to quite complicated expressions for $K_{\text{in}}'(t)_{1,2}$ and $K_{\text{out}}'(t)_{1,2}$. Equations (C24) and (C28) have been written in a manner which suggests the possibility for making an "approximate" first-order calculation of $\Delta_{01}^{(0)}$ by neglecting terms with more than one f_2 function. This approximation is also implicit when one uses the second line of (C5) for $\mathcal{K}''(t,s)$. Similarly, it might be possible to simplify Eqs. (C26) and (C27) in an approximate manner and, in fact, terms involving f_2 functions have already been omitted from these expressions for reasons of simplicity. Finally, we note that we have used the approximate formulas (43)–(45) of MIII for the line factors in the derivation of Eqs. (C28)–(C31) and this has involved the further neglect of f_2 functions. Unfortunately, until additional insight into these equations has been gained or numerical studies have been made, any such approximation cannot really be justified.

There is one good argument which suggests that the quantities of Eqs. (C28)–(C31) are indeed small for the general Bose fluid. Thus, when one studies the DHSBG limit by using Eqs. (C16a) and (C17a), which are valid for all T (such that $x>0$), then one finds that in first order $K_{\text{in}}'(t)_{1,2}=K_{\text{out}}'(t)_{1,2}=0$. This remarkable result, overlooked in MIII, means that to $O(a^{5/2})$

$$\Delta_{01}^{(0)}=0 \text{ (for the DHSBG).} \tag{C33}$$

Since this result is valid for all T, and not just $T=0$, one has reason to believe that $\Delta_{01}^{(0)}$ and the quantities of Eqs. (C28)–(C31) are probably small, even for He II. An approximate calculation of them would suffice.

(6) We now state an important identity which is extremely useful in the derivation of the expressions for $K_{\text{in}}'(t)_{1,2}$ and $K_{\text{out}}'(t)_{1,2}$ of step 5, Eqs. (63) and (64) of Sec. 5 for $\Delta_1^{(0)}$ and $\Delta_2^{(0)}$, respectively, and finally for $\Delta_{01}^{(0)}$:

$$\begin{aligned} \int_{t_1}^{t_2} ds e^{-sA}\{g_{ij}(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)+\sum_{\mathbf{k}_5\mathbf{k}_6} e^{(s-t)B_{ij}}f_2(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_5\mathbf{k}_6|\mathbf{k}_3\mathbf{k}_4)B_{ij}^{-1}\} \\ = -A^{-1}\{e^{-t_2A}G_{ij}^A(t_2-t,\,\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)-e^{-t_1A}G_{ij}^A(t_1-t,\,\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)\}, \end{aligned} \tag{C34}$$

where B_{ij} is given by Eq. (C19) and we have used (C20).

(7) We finally observe that some of the terms containing $[\exp s(\epsilon_+ +\epsilon_-)]$ as temperature-dependent factors in $K_{\text{in}}'(t)$ and $K_{\text{out}}'(t)$ [Eqs. (C26) and (C27)] become temperature-independent in the DHSBG limit, for in the latter case $\epsilon_+ +\epsilon_- =0$ to first order. In this limit these terms do not give a contribution to $\Delta_{01}^{(0)}$ but must be included in the $\Delta_2^{(0)}$ expression of Eq. (64). The contribution to $\Delta_2^{(0)}$ in this case is

$$\begin{gathered} -W(n\Omega)^{-1}\sum_{\mathbf{p}} \nu'F_-P\alpha^{-1}(\epsilon_+-\epsilon_-)^{-2}(\epsilon_++\epsilon_1)(\epsilon_-+\epsilon_1)\delta_{\epsilon_+,\epsilon_-}, \\ W=8\pi\hbar^2na/M. \end{gathered} \tag{C35}$$